Lecture 5A
MTH6102: Bayesian Statistical Methods

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Today's agenda

Today's lecture will

- Review
- Compute Bayes point estimates given a pmf or pdf posterior distribution.
- Construct credible intervals given a pmf or pdf posterior distribution.

Review: Bayesian updating

Bayesian updating: Using Bayes' theorem to update a prior distribution to a posterior distribution given data and the likelihood.

- Observed data y come from $p(y \mid \theta)$, where θ is unknown.
- Prior distribution, $p(\theta)$ of θ (pmf or pdf).
- Likelihood: $p(y \mid \theta)$ (discrete or continuous)

Bayes' theorem

$$p(\theta \mid y) = \frac{p(\theta)\, p(y \mid \theta)}{p(y)}$$

Posterior distribution $\propto$ prior distribution $\times$ likelihood

Review: Conjugate priors

- A prior is conjugate to a likelihood, $p(y \mid \theta)$, if the posterior is the same type of distribution as the prior.

- Advantage: Bayesian updating reduces to modifying the parameters of the prior distribution.

Review: Examples of likelihood/conjugate prior pairs

	hypothesis	data	prior	likelihood	posterior
Bernoulli/Beta	$\theta \in [0, 1]$	$x = 0$ or $x = 1$	Beta(α, β)	Bernoulli(θ)	Beta$(\alpha + 1, \beta)$ or Beta$(\alpha, \beta + 1)$
Binomial/Beta (fixed n)	$\theta \in [0, 1]$	$x = k$	Beta(α, β)	binomial(n, θ)	Beta$(\alpha + k, \beta + n - k)$
Geometric/Beta	$\theta \in [0, 1]$	$x = k$	Beta(α, β)	geometric(θ)	Beta$(\alpha + k, \beta + 1)$
Normal/Normal (fixed σ^2)	$\theta \in \mathbb{R}$	x	$N(\mu_0, \sigma_0^2)$	$N(\theta, \sigma^2)$	$N(\mu_1, \sigma_1^2)$
Normal/gamma (fixed θ)	$\tau = 1/\sigma^2 > 0$	$x \in \mathbb{R}$	gamma(α, β)	$N(\theta, \sigma^2)$	gamma$(\alpha + 0.5, \beta + 0.5(x - \theta)^2)$
Exponential/Gamma	$\lambda > 0$	$x > 0$	gamma(α, β)	exponential(λ)	gamma$(1 + \alpha, x + \beta)$

Board question

Which are conjugate priors for the following pairs likelihood/prior?

1. Exponential/Normal
2. Exponential/Gamma
3. Binomial/Normal

Solution

(1) Let $x \sim \text{exponential}(\theta)$. The likelihood is

$$p(x|\theta) = \theta e^{-\theta x}$$

$\theta \sim N(\mu_0, \sigma_0^2)$ so the prior is

$$p(\theta) = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{-\frac{(\theta-\mu_0)^2}{2\sigma_0^2}\right\}$$

The posterior density, $p(\theta|x)$, is

$$p(\theta|x) \propto \text{prior} \times \text{likelihood}$$

$$\propto \theta e^{-\theta x} \times \exp\left\{-\frac{(\theta-\mu_0)^2}{2\sigma_0^2}\right\}$$

$$= \theta \exp\left\{-\theta x - \frac{(\theta-\mu_0)^2}{2\sigma_0^2}\right\}.$$

The factor of θ before the exponential means that the posterior density is not normal. So, the normal is not conjugate prior for the exponential likelihood.

(2) Yes (see table).

(3) No, obviously

Normal example, both parameters unknown

- If μ and $\tau = 1/\sigma^2$ are unknown then there is a bivariate distribution which is conjugate.
- Marginal distribution

$$\tau \sim \text{Gamma}$$

and conditional distribution

$$\mu \mid \tau \sim \text{Normal.}$$

- The joint prior distribution is the product of these two.
- The posterior is of the same form.
- We're not going into details in this module.

Review from probability

- Let A and B two events.

Then
$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$ Multiplication rule

If A and B are independent, $P(A|B) = P(A)$, $P(B|A) = P(B)$

If X and Y are continuous RV, we have

$$f_{XY}(x,y) = f_{X|Y}(x|y) f_Y(y) = f_{Y|X}(y|x) f_X(x)$$ Multiplication rule for densities.

If X and Y are independent

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

because $f_{X|Y}(x|y) = f_X(x)$

In the normal example, our random variables are μ and $\tau = 1/\sigma^2$

By the multiplication rule

$p(\mu, \tau) = p(\mu|\tau)\, p(\tau)$

$p(\tau) \sim$ gamma

$p(\mu|\tau) \sim N(\mu_0, \sigma_0^2)$

My posterior density, $p(\underline{\mu, \tau}/y)$

$$\text{posterior} = p(\mu, \tau|y) \propto \text{prior} \times \text{likelihood}$$
$$\propto p(\mu, \tau) \times N(\mu, \sigma^2)$$
$$\propto p(\mu|\tau) \cdot p(\tau) \times N(\mu, \sigma^2)$$

because data $\sim N(\mu, \sigma^2)$

Problem 2, ex. sheet 4

You have parameters θ and ψ which are independent under the prior. So the joint prior of θ and ψ is

$$p(\theta, \psi) = p(\theta) p(\psi)$$

You need to show that θ and ψ are still independent under the posterior:

$$p(\theta, \psi \mid x) = \underline{p(\theta \mid x)}\, \underline{p(\psi \mid x)}$$

Need to show this

$$p(\theta, \psi \mid x) \propto p(\theta, \psi) \times \text{likelihood}$$

$$= \underline{p(\theta)}\, \underline{p(\psi)} \times \underline{\text{likelihood}}$$

Bayesian inference

- Data y come from $p(y \mid \theta)$, where θ is unknown.
- We have seen how to calculate the posterior distribution for parameter θ by

$$p(\theta \mid y) \propto p(\theta)\, p(y \mid \theta)$$

Posterior distribution $\propto$ prior distribution $\times$ likelihood

- In the Bayesian framework, all our inferences about θ are based on the posterior distribution $p(\theta \mid y)$.
- This includes point estimates.
- For a single parameter, we can summarize the posterior distribution just as we would normally summarize a distribution.

Point estimates

- Suppose we know the posterior distribution $p(\theta \mid y)$ for a one-dimensional parameter θ.
- We could summarise the center of the posterior $p(\theta \mid y)$ using e.g.,
  - mean
  - median
  - mode
- Mean or median are most common.
- Mode may be used if it's difficult to calculate mean or median.

Point estimates

$EX = \int x f_X(x) dx$

Summaries of $p(\theta \mid y)$ as point estimates for θ.

- Posterior mean, for a pdf posterior density

$$\hat{\theta}_{\mathrm{B}} = \int_\theta \theta p(\theta \mid y)\, d\theta$$

- Median, $\hat{\theta}_m$

$$P(\theta \leq \hat{\theta}_m | y) = 0.5.$$

- Mode or maximum a posteriori (MAP)

$$\hat{\theta}_{\mathrm{MAP}} = \operatorname{argmax}_\theta p(\theta \mid y).$$

function of θ

Point estimates for Beta posterior pdf

Beta$(k+\alpha, n-k+\beta)$ posterior distribution.

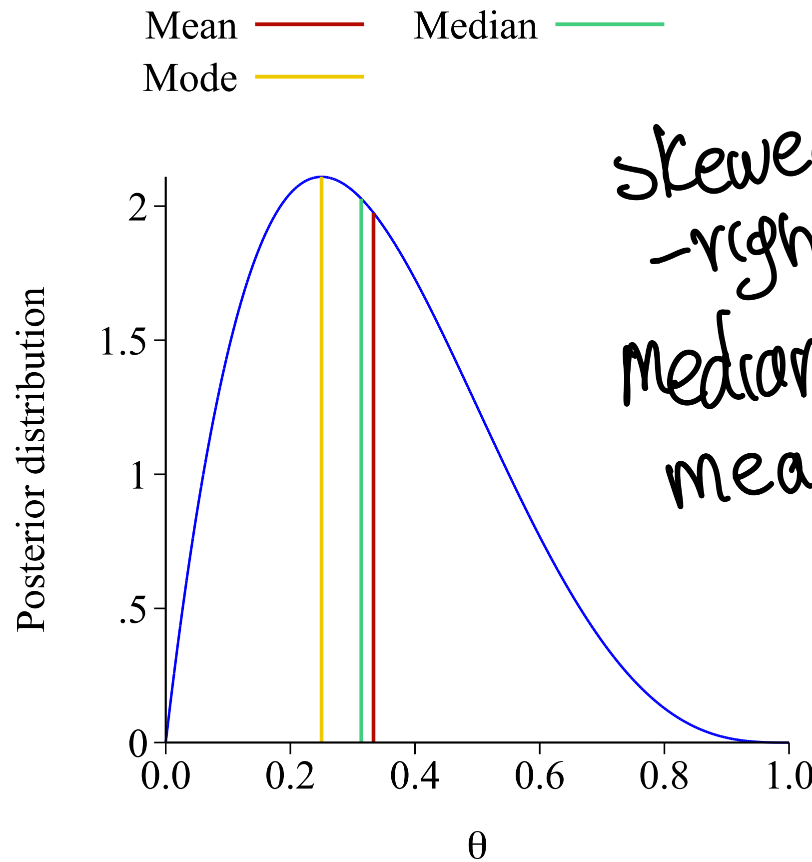
- Mean:

$$\frac{k+\alpha}{n+\alpha+\beta}$$

- Mode:

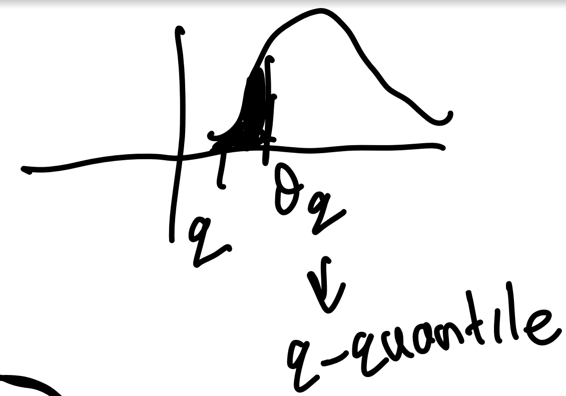
$$\frac{k+\alpha-1}{n+\alpha+\beta-2}$$

- No simple formula for median but we can use computer.



skewed-right
median < mean

Quantile function



- For a RV Θ, let $F(\theta)$ be the cdf

$$P(\Theta \leq \theta) = F(\theta)$$

- If F is strictly increasing and continuous, then $F^{-1}(q)$, $q \in (0,1)$ is the unique real number θ_q such that

$$F(\theta_q) = q$$

- We call θ_q the q-quantile of Θ.
- The quantile function is the inverse function of the cdf

$$Q = F^{-1}$$

- If $q = F(\theta_q)$ for some $q \in (0,1)$, then $Q(q) = \theta_q$.

Quantile function

- E.g. if $q = 0.5$ and $m = \theta_{0.5} = F^{-1}(1/2)$ is the median,

$$F(\theta_{0.5}) = 0.5$$

$$Q(0.5) = \theta_{0.5}$$

- We call $F^{-1}(1/4)$ the first quantile and $F^{-1}(3/4)$ the third quantile.

$F^{-1}(1/4) = \theta_{1/4}$

$F^{-1}(3/4) = \theta_{3/4}$

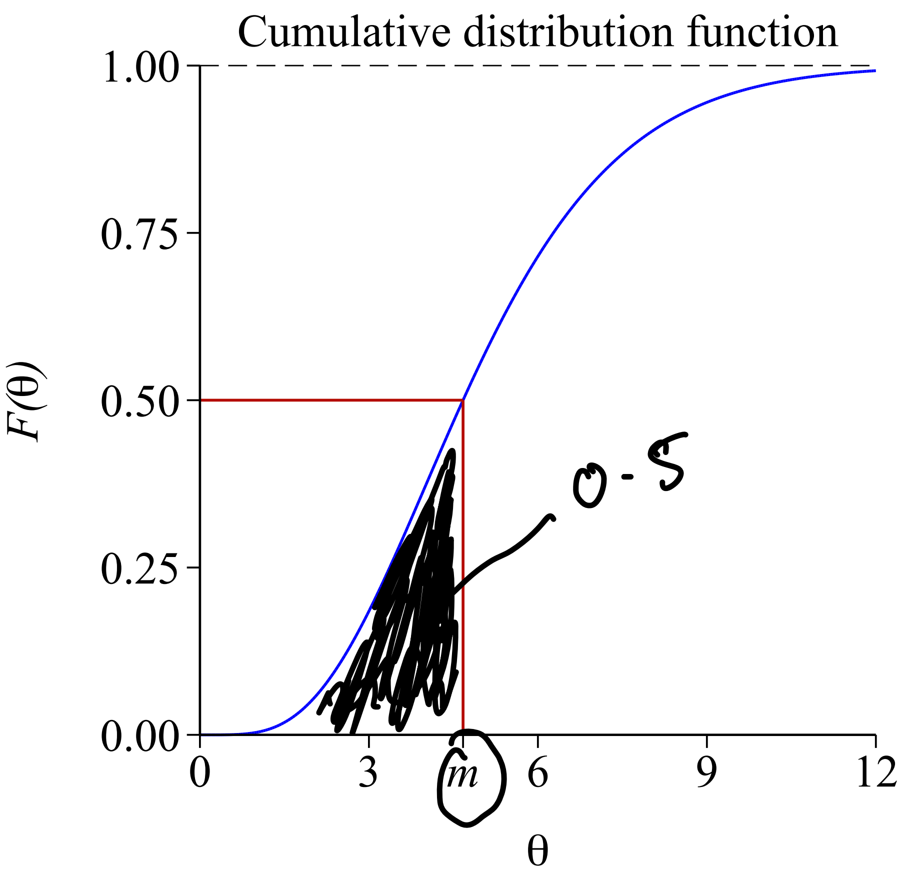
Finding the median

- Let $F(\theta)$ be the cdf

$$P(\Theta \leq \theta) = F(\theta)$$

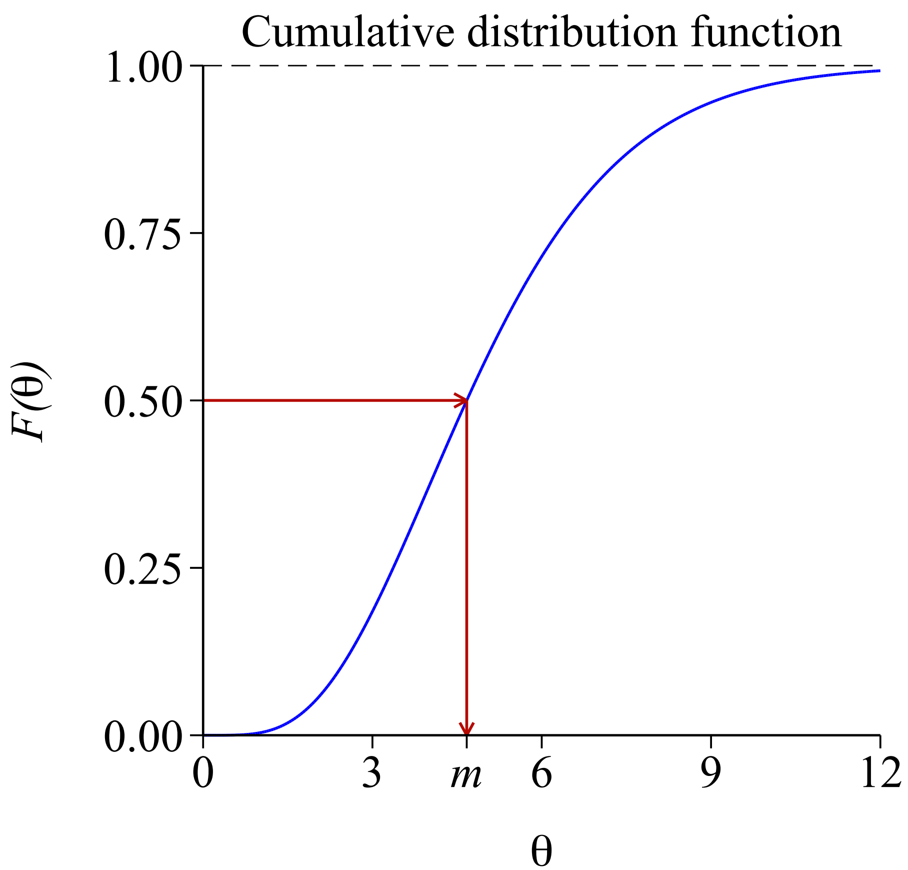
- If m is the median, then $F(m) = 0.5$.
- Half the probability mass is below, and half is above

$$P(\Theta \leq m) = 0.5$$



Finding the median

- So if we can find the inverse function of the cdf, we can find the median.
- The inverse of the cdf is called the quantile function.



Finding the median

- We have seen examples where the posterior distribution is in a well-known family of distributions.
- E.g. beta, gamma, or normal.
- Each one has a simple formula for the mean.
- For beta or gamma, there is no direct formula for the median (or the cdf).
- But we can use functions in R.
- E.g. for the gamma distribution `pgamma` returns the cdf and `qgamma` returns the quantile function (inverse of cdf).

Board question

Bent coin with unknown probability θ.
Flat prior: $p(\theta) = 1$ on $[0, 1]$
Data: toss 27 times and get 15 heads.

1. Find the posterior mean
2. Find the posterior median.
3. Find the MAP. /mode

Solution

This is a Binomial/beta conjugate prior example.
The posterior, $p(\theta|x)$, is

$$p(\theta|x) \propto \text{prior} \times \text{likelihood}$$
$$\propto p(\theta) \times p(x|\theta)$$

- $p(\theta) = 1 \quad \forall \text{ all } \theta \in [0,1]$
- $p(x|\theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$

Thus,

$$p(\theta|x) \propto 1 \times \theta^x (1-\theta)^{n-x}$$
$$= \theta^x (1-\theta)^{n-x}$$
$$= \theta^{(x+1)-1} \cdot (1-\theta)^{(n-x+1)-1}$$

α ... β

$$\theta^{\alpha-1} (1-\theta)^{\beta-1}$$

Beta(α, β)

Beta$(x+1, n-x+1)$

$x = 15$, $n = 27$

$$p(\theta|15) \sim \text{Beta}(16, 13)$$

In our case,

The posterior mean is

- $\hat{\theta}_B = \frac{16}{16+13} = \frac{16}{29}$

The mode / MAP of θ is

- $\hat{\theta}_{MAP} = \frac{15+1-1}{27+1+1-2} = \frac{15}{27}$

- The median, $\hat{\theta}_{0.5}$, is found by

$$P(\theta \le \hat{\theta}_{0.5}) = \int_{-\infty}^{\theta_{0.5}} p(\theta|x)\, d\theta = 0.5$$

→ use computer

- The MLE of θ is $\hat{\theta} = \frac{x}{n} = \frac{15}{27}$

The MLE of θ and the MAP are identical because

$p(\theta|x) \propto$ binomial likelihood

Maximizing $p(\theta|x)$ over θ is equivalent to maximizing the binomial likelihood over θ

The posterior density is proportional to the binomial likelihood. ⇒ they have the same shape.

the binomial likelihood when n is large becomes more symmetric and peaked around the MLE.

Uncertainty in parameters

- In Bayesian inference, any statements about uncertainty are based on the posterior distribution $p(\theta \mid y)$.
- For a single summary of uncertainty, we can calculate the posterior standard deviation.
- This is just the square root of the variance of the distribution.
- For example, for the beta($\alpha + k$, $\beta + n - k$) pdf, the posterior variance of θ is

$$\mathrm{var}(\theta \mid k) = \frac{(\alpha + k)(\beta + n - k)}{(\alpha + \beta + n)^2(\alpha + \beta + n + 1)}.$$

Confidence intervals

In frequentist inference (i.e. non-Bayesian inference), confidence intervals are used to express a range of uncertainty around a parameter estimate.

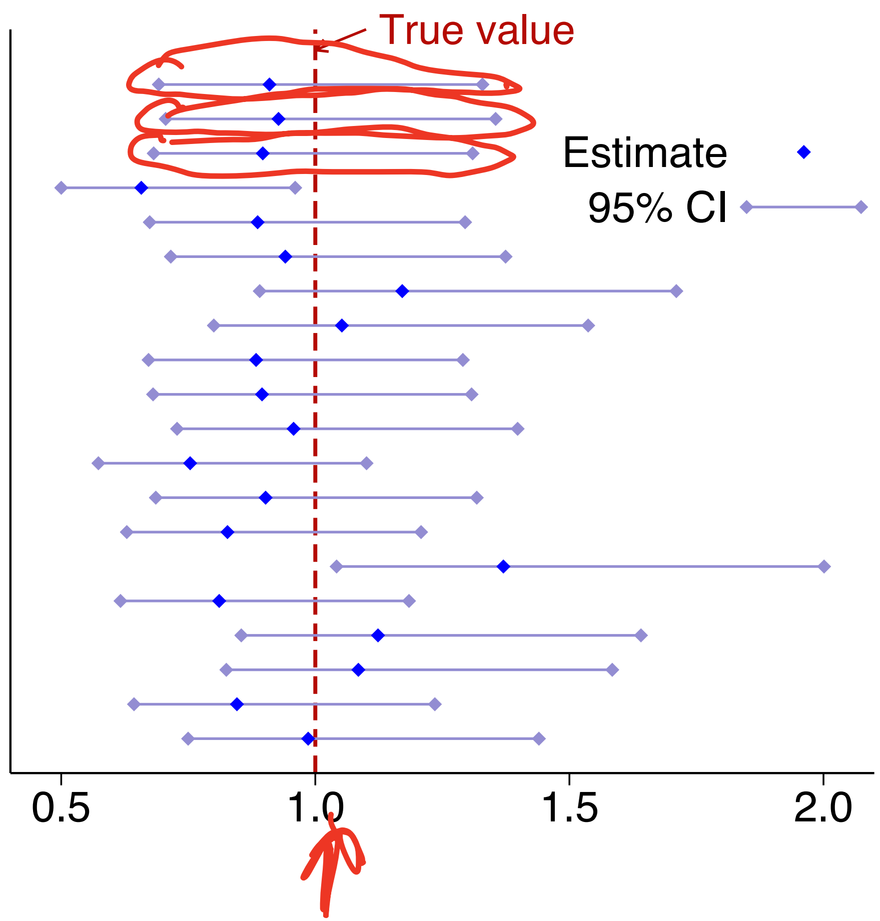
- Suppose random samples $Y = (Y_1, \ldots, Y_n)$ are repeatedly generated.
- For each sample we can estimate the true parameter θ by $\hat{\theta}(Y)$, and also construct an interval estimator $(\theta_L(Y), \theta_U(Y))$ based on the random sample $Y = (Y_1, \ldots, Y_n)$.
- A 95% confidence interval is an interval $(\theta_L(Y), \theta_U(Y))$ that covers θ with probability 0.95

$$P\left(\theta_L(Y) \leq \theta \leq \theta_U(Y)\right) = 0.95$$

- The probability 0.95 refers to the random interval $(\theta_L(Y), \theta_U(Y))$, and not the parameter and is called the coverage probability.

Confidence intervals illustrated

- Generate repeated samples from some distribution.
- Estimate $\hat{\theta}$ and a 95% confidence interval for $\hat{\theta}$ each time.
- 95% of the random intervals should contain the true value.



Interpretation of confidence intervals

- In classical statistics, it is NOT correct to say θ lies in the interval $(\theta_L(y), \theta_U(y))$ with probability 0.95 since θ is assumed to be fixed.

- The interval $(\theta_L(y), \theta_U(y))$ is one of the possible realised values of the random interval $(\theta_L(Y), \theta_U(Y))$ when $Y = y$, and since θ is fixed, θ is in $(\theta_L(y), \theta_U(y))$ with probability 0 or 1.

- **Long-run frequency interpretation.** With frequentist confidence intervals, when we say that the interval $(\theta_L(y), \theta_U(y))$ has 0.95 chance of coverage we only mean that, in the long run, with repeated sampling, the intervals trap the parameter θ 95% of the time.

Credible intervals

- In the Bayesian framework, we can say that θ lies inside the interval with some probability, not 0 or 1.
- θ is a random variable with a probability distribution.
- After seeing the data y, this is the posterior distribution

$$p(\theta \mid y).$$

- As well as summarizing the posterior with a point estimate, we can directly calculate an interval for θ using the posterior distribution.
- They are called credible intervals or probability intervals.

Credible intervals

- For some $\alpha \in [0, 1]$, a $100(1-\alpha)\%$ credible or probability interval for θ is an interval (θ_L, θ_U) such that

$$P(\theta_L < \theta < \theta_U) = 1 - \alpha$$

Probability statement is about θ.

  E.g. $\alpha = 0.05$ for a 95% credible interval.

- More generally, (θ_L, θ_U) is a p-probability or credible interval for θ such that

$$P(\theta_L < \theta < \theta_U) = p$$

- The probabilities are calculated from the posterior distribution pmf or pdf

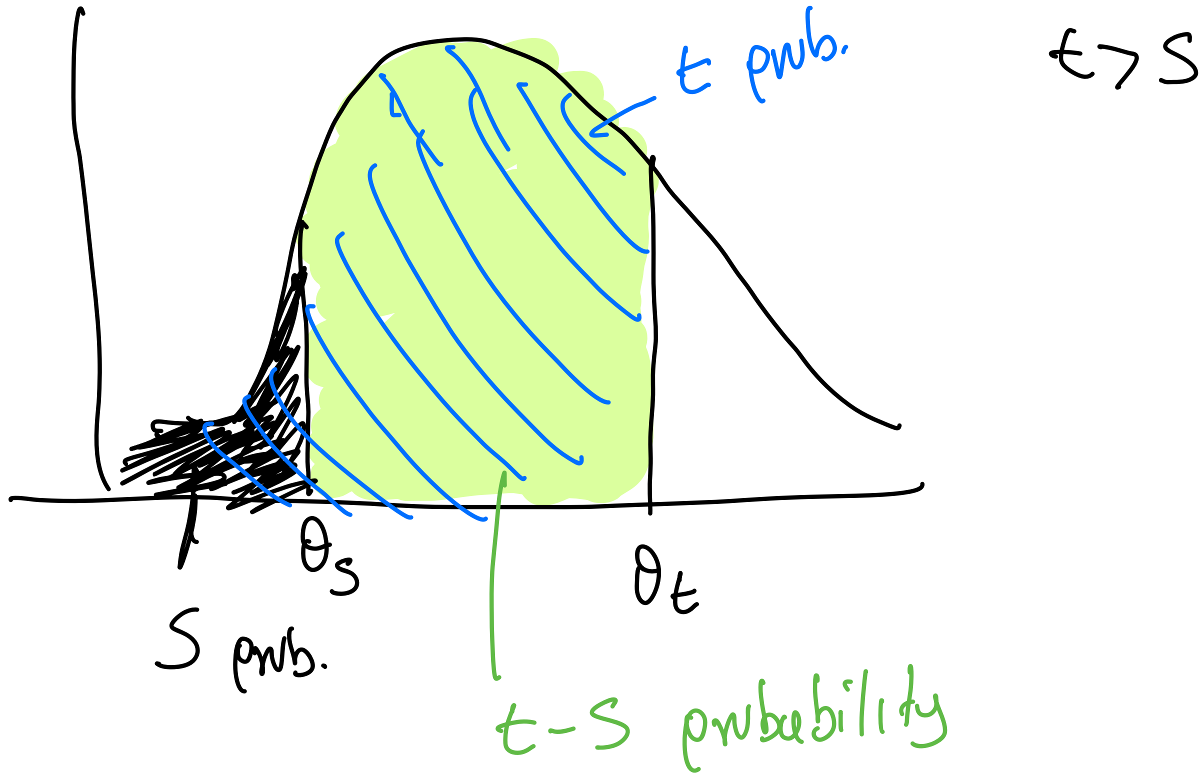
$$p(\theta \mid y)$$

$$P(\theta \in [\theta_L, \theta_U]) = \int_{\theta_L}^{\theta_U} p(\theta \mid y)\, d\theta = p$$

Credible intervals

- There are many ways to compute a p-credible interval.
- In particular, notice that the p-credible interval for θ is not unique.
- **Example:** Between the 0.05 and 0.55 quantiles is a 0.5 probability interval. Another 0.5-probability interval goes from 0.25 to the 0.75 quantiles.
- Thus we have 0.5 probability intervals $[\theta_{0.05}, \theta_{0.55}]$ and $[\theta_{0.25}, \theta_{0.75}]$.

$p = 0.5$



t prob.
t > S
θs
θt
S prob.
t - S probability

Equal tail intervals or symmetric probability intervals

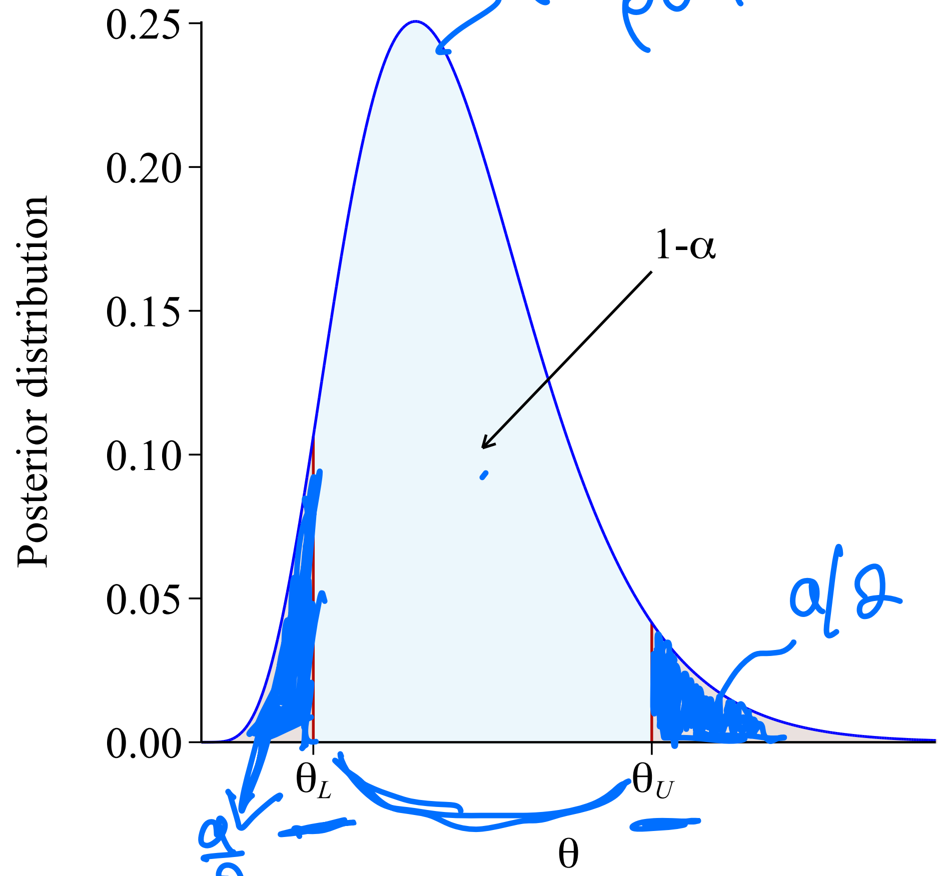
- Posterior pdf shown.
- $100(1-\alpha)\%$ interval.

$$P(\theta_L < \theta < \theta_U) = 1 - \alpha$$

- Equal probability outside each end.

$$P(\theta < \theta_L) = \alpha/2$$
$$P(\theta > \theta_U) = \alpha/2$$

- **Example:** If $\alpha = 0.5$, the interval $[\theta_{0.25}, \theta_{0.75}]$ is symmetric because the amount of probability remaining on either side of the interval is the same, namely 0.25.



[θ0.25, θ0.75] is a 95% -equal-tail interval

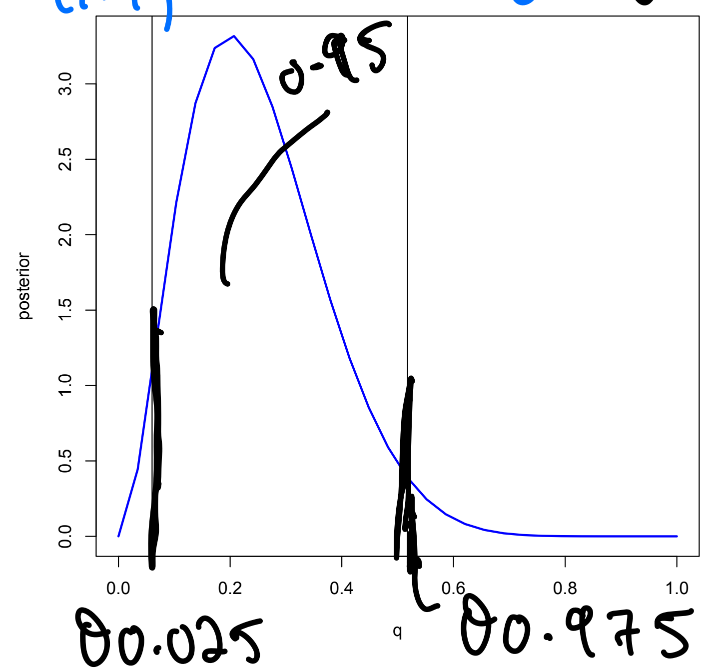
Board question: beta credible interval

Bent coin with unknown probability θ.
Flat prior: $p(\theta) = 1$ on $[0, 1]$ ~Beta(1,1)
Data: toss 10 times and get 2 heads.
n=10 x=2

p(θ|x) ~ Beta(1+x, 1+n−x) = Beta(3, 9)

1. Use R to construct a symmetric 95% credible interval
2. `qbeta(c(0.025,0.975),shape1=3,shape2=9)`

θL = F⁻¹(1/4) θU = θ0.975 0.975 − 0.025 = 0.95



3. A beta(3,9) posterior distribution with vertical bars indicating a 95% probability interval.

Board question: Normal credible set

Let $x_1, \ldots, x_n$ an i.i.d from $N(\theta, \sigma^2)$ where σ^2 is known. Let θ have prior $N(\mu, \tau^2)$, where μ and τ are known.

- Find a $1 - \alpha$ credible interval for θ.

Solution: Normal credible interval

$x_1, \dots, x_n \sim N(\theta, \sigma^2)$, σ^2 known

$\theta \sim N(\mu, \tau^2)$

We know that $p(\theta | x_1, \dots, x_n) \sim N(\mu_1, \sigma_1^2)$, where

$$\mu_1 = \frac{a\mu + b\bar{x}}{a+b}, \qquad a = \frac{1}{\tau^2}$$

$$\sigma_1^2 = \frac{1}{a+b}, \qquad b = \frac{n}{\sigma^2}$$

We want to find $(1-a)\%$ credible interval, $[\theta_L, \theta_U]$ for θ such that

$$\underbrace{P(\theta_L \le \theta \le \theta_U)}_{\text{based on the posterior!}} = 1-a, \quad \theta \sim N(\mu_1, \sigma_1^2)$$

We know

$$P(\theta_L \le \theta \le \theta_U)$$

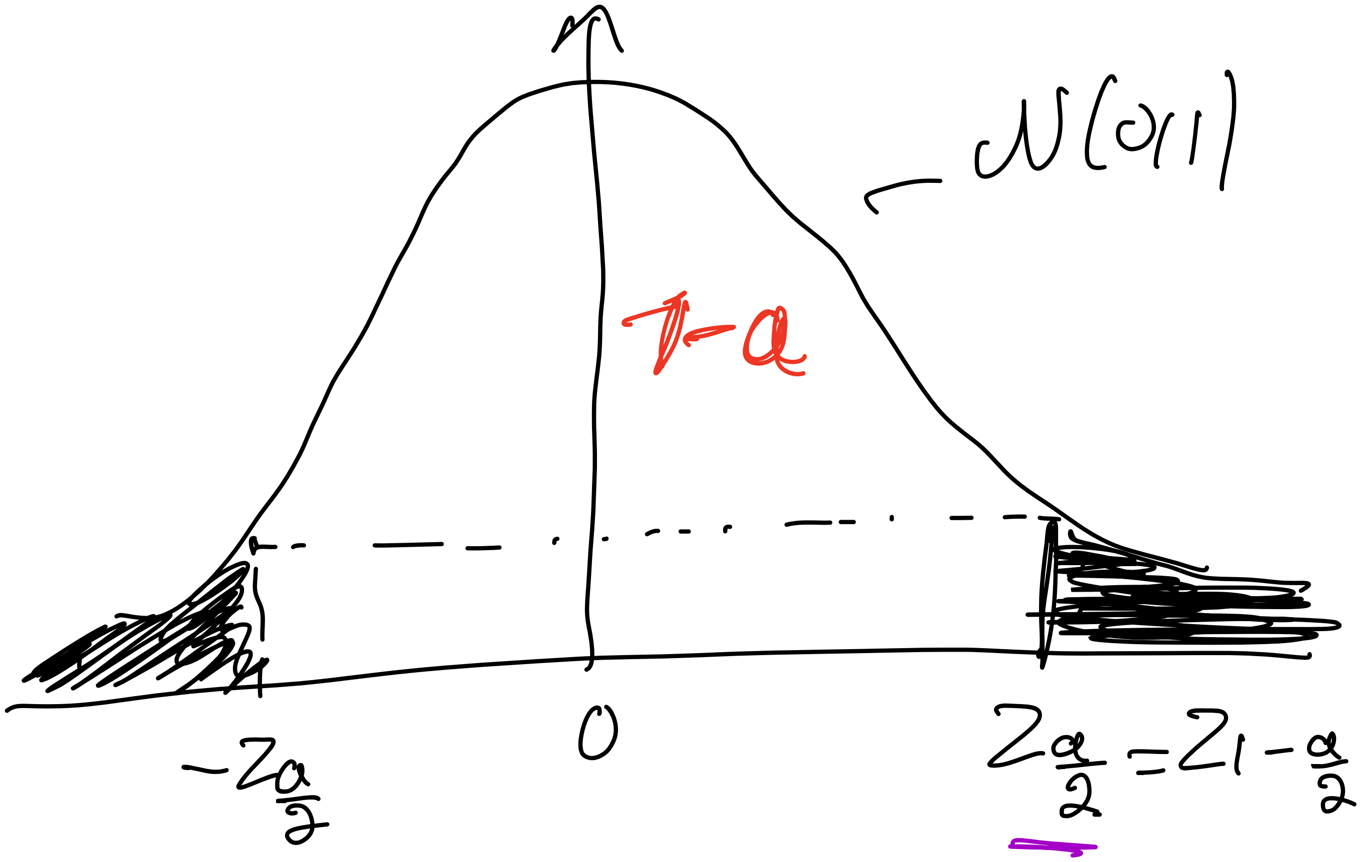
$$= P\left(\frac{\theta_L - \mu_1}{\sqrt{\sigma_1^2}} \le \frac{\theta - \mu_1}{\sqrt{\sigma_1^2}} \le \frac{\theta_U - \mu_1}{\sqrt{\sigma_1^2}}\right) = 1-a$$

and

$$\frac{\theta - \mu_1}{\sqrt{\sigma_1^2}} \sim N(0,1)$$

We know if $Z \sim N(0,1)$

$$P\left(-Z_{\frac{\alpha}{2}} < Z \le Z_{\frac{\alpha}{2}}\right) = 1-\alpha$$



Thus, we can take

$$\frac{\theta_L - \mu_1}{\sqrt{\sigma_1^2}} = -Z_{\frac{\alpha}{2}}$$

$$\frac{\theta_U - \mu_1}{\sqrt{\sigma_1^2}} = Z_{\frac{\alpha}{2}}$$

} Solve for θ_L and θ_U

To find -

$$\theta_L = \mu_1 - Z_{\frac{a}{2}} \sqrt{\sigma_1^2}$$

$$\theta_U = \mu_1 + Z_{\frac{a}{2}} \sqrt{\sigma_1^2}$$

Thus, a normal $(1-a)\%$ credible interval for θ is

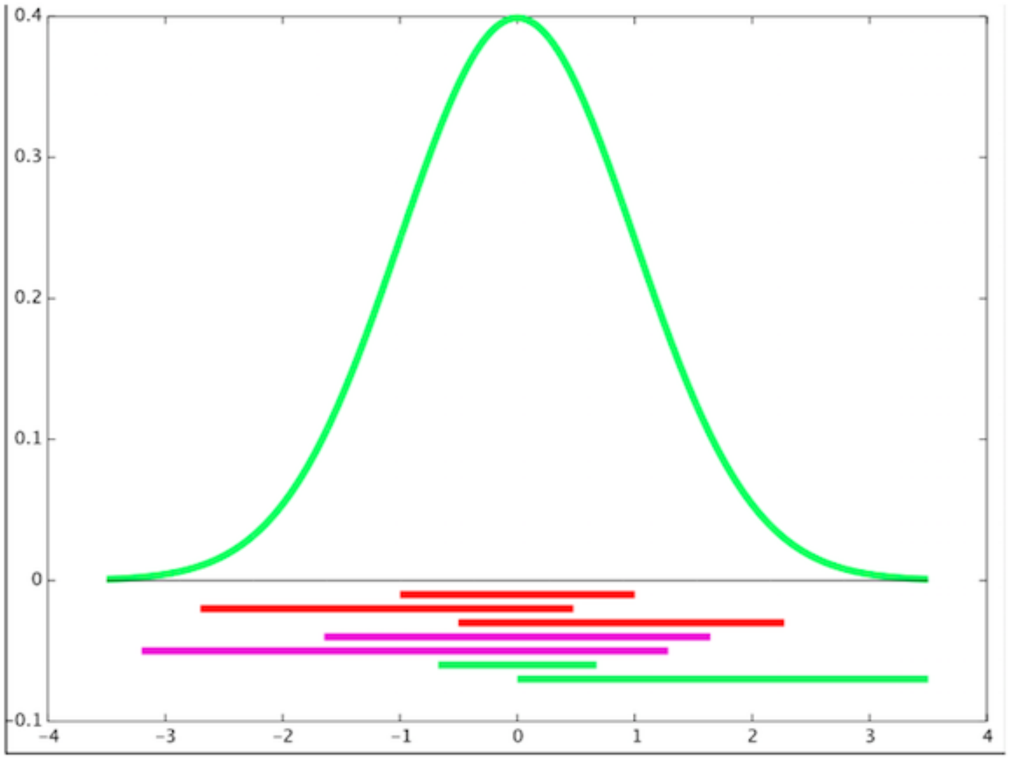
$$\left[\mu_1 - Z_{\frac{a}{2}} \sigma_1 , \mu_1 + Z_{\frac{a}{2}} \sigma_1\right]$$

if $a = 0.05 \Rightarrow 95\%$ credible interval

$$\boxed{\mu_1 \pm 1.96 \sigma_1}$$

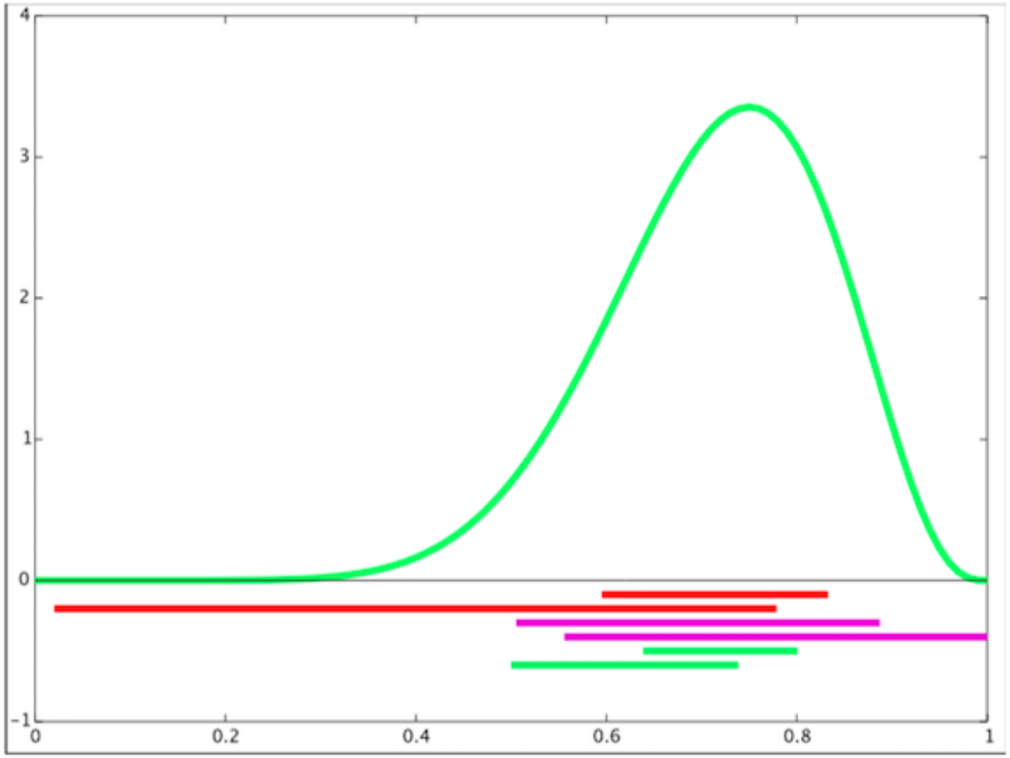
$Z_{\frac{a}{2}} = 1.96$

Probability intervals for beta distributions



Red = 0.68, magenta = 0.9, green = 0.5

Probability intervals for normal distributions



Red = 0.68, magenta = 0.9, green = 0.5

Credible intervals

Remarks

- For a fixed, p, different p-credible intervals for θ may have different $widths$.
- Since the width can vary for fixed p, a larger p does now always mean a larger width. But if a p_1-credible interval is fully contained in a p_2-credible interval, then p_1 is smaller than p_2.
- As in classical statistics, we can obtain a smallest credible interval by centering the interval under the highest part of the pdf posterior. Such an interval is called highest posterior density interval and is usually a good choice since it contains the most likely values.

Board question

To convert an 80% probability interval to a 90% interval should you shrink it or stretch it?

1. Shrink.
2. Stretch.

Highest posterior density (HPD) intervals

- If the posterior density $p(\theta|y)$ is unimodal, then for a given values of α, the 1- α- shortest credible interval for θ is given by

$$\{\theta : p(\theta|y) \geq k\},$$

where k is chosen so that

$$\int_{\{\theta : p(\theta|y) \geq k\}} p(\theta|y)\, d\theta = 1 - \alpha.$$

- The set $\{\theta : p(\theta|y) \geq k\}$ is called the highest posterior density (HPD) interval, as it consists of the values of the parameter θ for which the posterior density is highest.

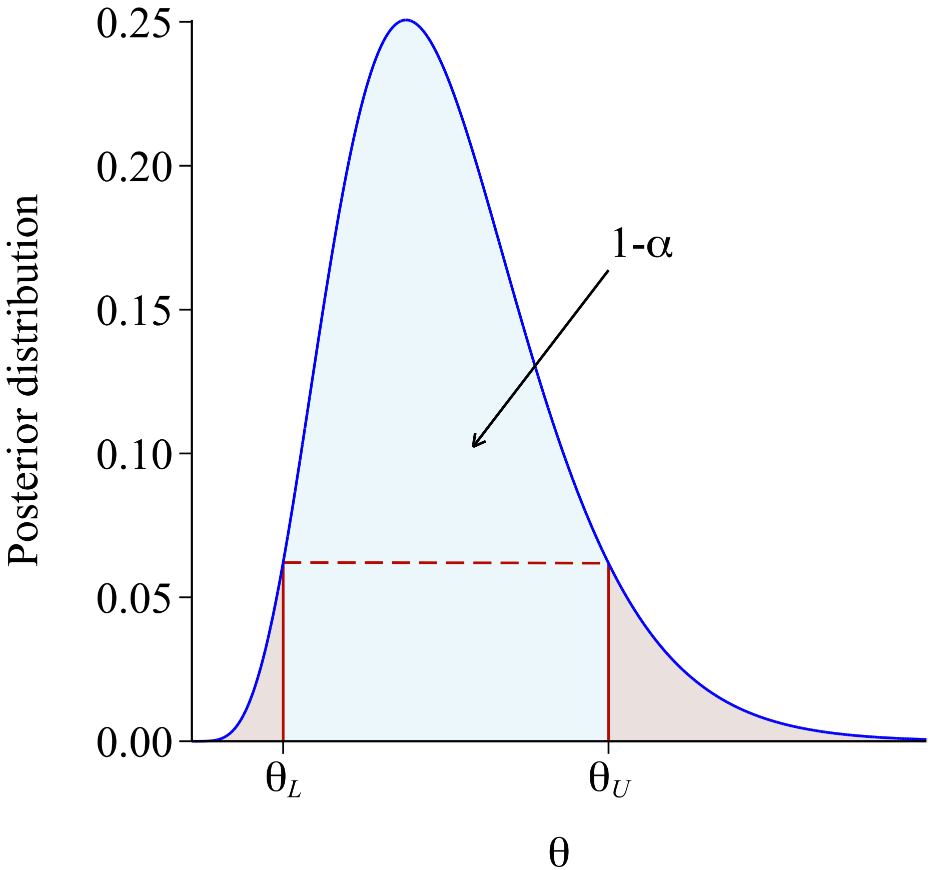
Highest posterior density (HPD) intervals

- Posterior pdf shown. We need to find θ_L and θ_U
- $100(1-\alpha)\%$ interval.

$$P(\theta_L < \theta < \theta_U) = 1 - \alpha$$

- Equal height to posterior density at θ_L and θ_U.

$$p(\theta_L \mid y) = p(\theta_U \mid y)$$



Calculating credible intervals

- Some textbooks emphasise the highest posterior density interval.
- However, it is usually difficult to calculate.
- The equal tail interval is easier to find computationally.
- For named distributions, just like for the median, we can use the quantile functions in R, `qgamma`, `qnorm` etc.

Suppose our posterior distribution for θ is Gamma(a, b).

Posterior median:

```
qgamma(0.5, shape=a, rate=b)
```

Equal tail 95% credible interval limits:

```
qgamma(0.025, shape=a, rate=b)
qgamma(0.975, shape=a, rate=b)
```