DYNAMICAL SYSTEMS

MTH744 U/P

SEMESTER A

2023 – 2024

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WEEKS

5, 6, 7, 8, 9

WS.1

3. Dynamics on the Circle S^1

$\mathbb{R}^1$

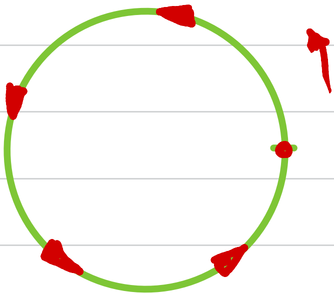
S^1

$\dot{\theta} = 1$

$\dot{\theta} = \theta^2$

$\dot{\theta} = f(\theta)$

needs $f(\theta + 2\pi) = f(\theta)$



$\theta = 0$ FPs

$\dot{\theta} = 1 - \cos\theta$

$\dot{\theta} = 0$

i.e. $1 - \cos\theta = 0$

$\theta = 0$ FP

note $\dot{\theta} > 0, \; \theta \neq 0$

Dynamics system on the circle

representing vf $f(x)$ as a graph $gr(f)$

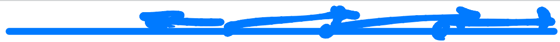
$$\frac{d\theta}{dt} = \dot{\theta} = f(\theta), \quad \theta \in S^1$$

cf. $\dot{x} = f(x)$

avoids overlaying vectors horizontally

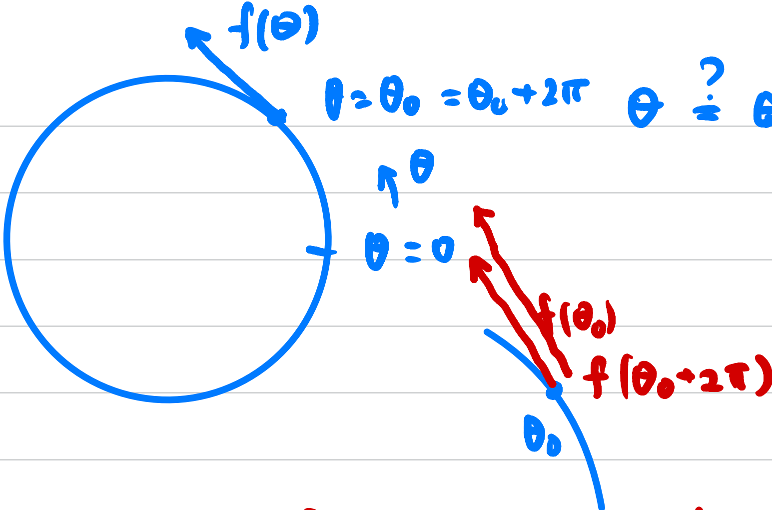
$\dot{x} = f(x)$

$gr(f)$



For $\dot{\theta} = f(\theta)$, we need periodic f e.g. "$\sin 3\theta$"

w5.2



$\theta \overset{?}{=} \theta + 2\pi \overset{?}{=} \theta + 2n\pi$

$n \in \mathbb{Z}$

We need f to be periodic in θ

i.e. $f(\theta) = f(\theta + 2k\pi)$, $k \in \mathbb{Z}$

W5.3

Fourier series

$$f(\theta) = \sum_{k \in I} a_k \cos(kx) + b_k \sin(kx)$$ p18

$f(\theta) = \ \ldots$ periodicity 2π

$I \subseteq \mathbb{Z}$

Note $k = 0$

$f(\theta) = a_0.$ ✓

Ex 3.1

$\dot{\theta} = w \rightarrow$ uniform motion

$\frac{d\theta}{dt} = w \qquad \theta = wt + \theta_0 \quad , \quad t = 0, \ \theta = \theta_0$

constant speed
not constant vel.

cf. $\dot{x} = w$ on $\mathbb{R}$

const speed & velocity

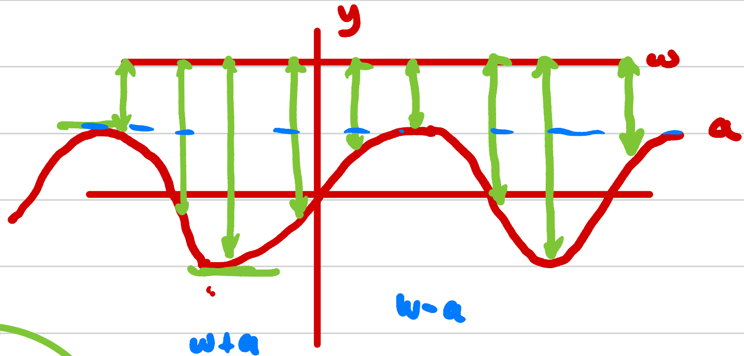
W5.4

$\dot{\theta} = \omega - a\sin\theta.$

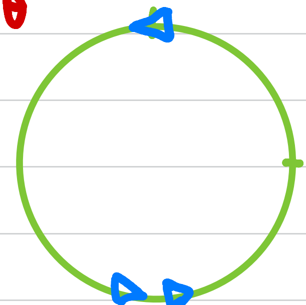
θ measured mod 2π

$\omega, a > 0$

$\omega > a > 0$



$\omega - a\sin\theta$



at $\theta = \pi/2$ as $a \to \omega$

speed slows to zero

$\theta = -\pi/2$

speed increases to 2ω

W5.5

What happens as $a \to \omega^-$ Slows down $\pi/2$

Speeds up at $-\pi/2$

$a = \omega$

$\dot\theta = \omega(1 - \sin\theta)$ $\exists$ fixed point(s)

$\sin\theta = 1$, $\theta = \pi/2$

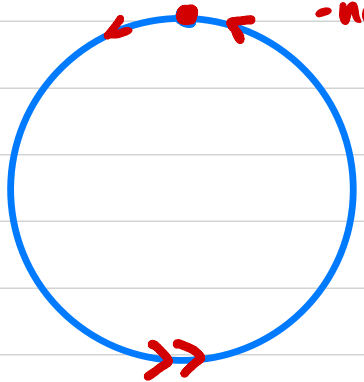
Saddle-node

$\dot\theta > 0$ $\theta \neq \pi/2$

$$\int dt = \int_{\theta_0}^{\theta_0+2\pi} \frac{d\theta}{\omega - a\sin\theta}$$

∞

p19



w5.6

As $a \to w^-$, then T for a complete circle to be covered by an orbit (solution are) tends to ∞.

As $a \to w^-$, a speed bottle-neck is created around $\theta = \pi/2$, and when we take the limit of $w = a$, a fixed point and a complete circuit no longer happens

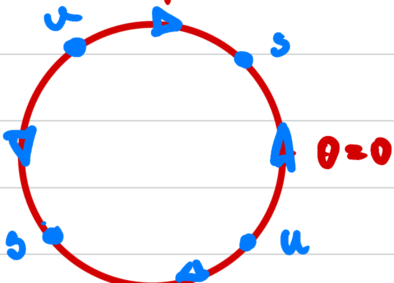
w5.7

p19 $\quad T = \dfrac{2\pi}{\sqrt{w^2 - a^2}} \quad \Rightarrow \quad a \to w^- \quad T \to \infty$

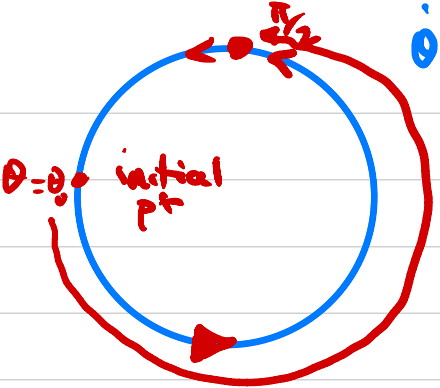
Ex $\quad \dot{\theta} = \cos 2\theta \qquad$ FPs $\quad \cos 2\theta = 0$

$2\theta = \pi/2 \,,\; -\dfrac{\pi}{2} \,,\; \dfrac{5\pi}{2} \,,\; +\dfrac{3\pi}{2}$

$\theta = \dfrac{\pi}{4} \,,\; -\dfrac{\pi}{4} \,,\; \dfrac{5\pi}{4} \,,\; \dfrac{3\pi}{4}$



w5.8



$\dot{\theta} = \omega(1 - \sin\theta)$

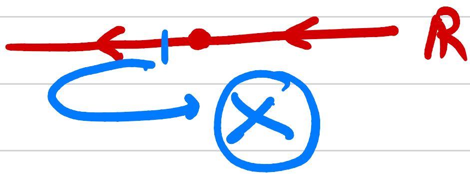
what set of points on S^1 are attracted to $\theta = \pi/2$

$\{\theta_0 \mid \theta(t)$ with $\theta(0) = \theta_0$ $\to \pi/2$ as $t \nearrow \infty\}$

Basin of attraction of the fixed pt $\theta = \pi/2$

$B\left(\frac{\pi}{2}\right) = S^1$

cf.



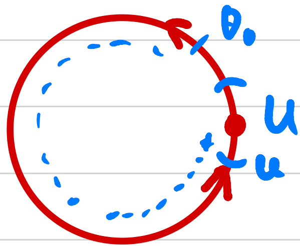
These examples show similarities with real line dynamical systems, but there are also differences

$\mathbb{R}$

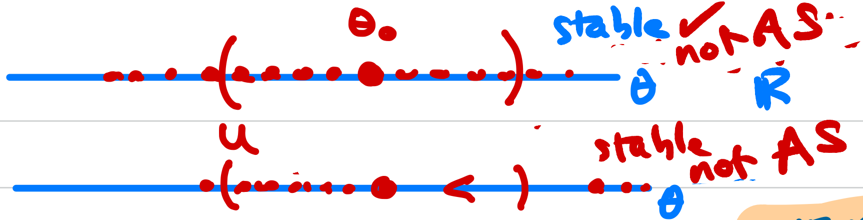
Start in U, nbd of x^*
we stay in U
$\to x_0$

x_0 is a stable fixed pt. Let $x_1 \in U$
then $x(t) \to x_0$, where $x(0) = x_1$; Also A.S. $\left(x(t) \to x_0 \ \forall x_1 \in U\right)$

S^1



start in U
sometimes we leave and then RE-ENTER!
$\to \theta_0$



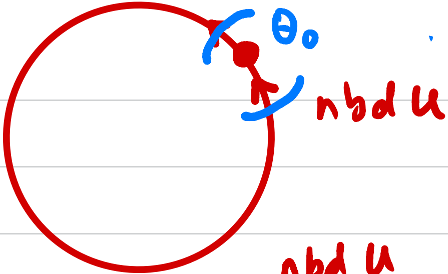
W5.10

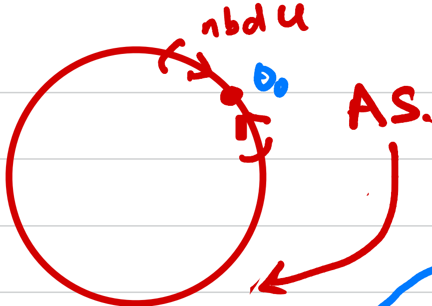
- stability on S^1 at fp $\theta = \theta_0$.

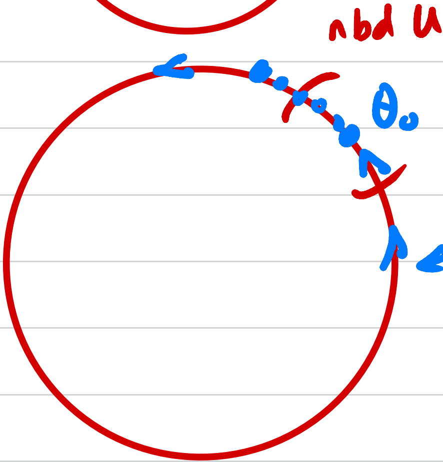
  $\exists$ U s.t. if $\theta_1 \in U$, then orbit $\theta(t)$ with $\theta(0) = \theta_1$ remains in U as $t \to \infty$

  asymptotic stability if $\theta_1 \in U$, $\theta(t)^* \to \theta_0$ as $t \to \infty$.

W5.11







S, not AS

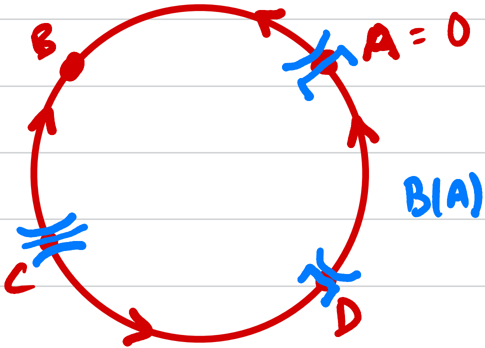
$\theta = \theta_0$ is a fixed pt in the circle phase portraits

S = stable

AS = asymptotically stable

Example of

Basins of attraction



$$\bigcup_{x \in \{A,B,C,D\}} B(x) = S^1$$

Basins of attraction partition the statespace S.

$\dot{x} = x$; $B(0) = \emptyset$, $B(\infty) = (0,\infty)$, $B(-\infty) = (-\infty, 0)$

Note for $\mathbb{R}$, we need to accommodate infinity

A few calculations on time to rotate a full circle

W5.13

$$\dot{\theta} = \omega - a\sin\theta \quad \Rightarrow \int \frac{d\theta}{\omega - a\sin\theta} = \int dt$$

$$u = \tan\left(\frac{\theta}{2}\right) \Rightarrow du = \sec^2\theta \, d\theta \Rightarrow du = \frac{(1+u^2)\,d\theta}{2}$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\theta,\ \cos^2\frac{\theta}{2} = \frac{2u}{1+u^2}$$

$$\int \frac{d\theta}{\omega - a\sin(\theta)} = \int \frac{2\,du}{(1+u^2)\left(\omega - \frac{a2u}{1+u^2}\right)} = \int \frac{2\,du}{\omega(1+u^2) - 2au}$$

$$= \frac{1}{\omega}\int \frac{2\,du}{u^2 - \frac{2au}{\omega} + 1} = \frac{1}{\omega}\int \frac{2\,du}{\left(u - \frac{a}{\omega}\right)^2 + \left(1 - \frac{a^2}{\omega^2}\right)} =$$

$$\frac{2}{w}\frac{1}{\sqrt{1-\frac{a^2}{w^2}}}\tan^{-1}\left(\frac{u-\frac{a}{w}}{\sqrt{1-\frac{a^2}{w}}}\right)$$

$$\left[\begin{array}{l}\theta=\pi\ ,\ u=\tan\frac{\pi}{2}=\infty\ ' \\ \theta=-\frac{\pi}{2}\ ,\ u=-\tan\frac{\pi}{2}=-\infty\end{array}\right]$$

W5.14

$$\frac{2}{\sqrt{w^2-a^2}}\left[\tan^{-1}\left(\frac{u-a/w}{\sqrt{1-a^2/w^2}}\right)\right]_{-\infty}^{\infty}=\frac{\pi}{\sqrt{w^2-a^2}}$$

See Latex notes if interested.

Dynamical System on $\mathbb{R}^2$ (introduction)

Ch 4

Linear Systems on $\mathbb{R}^2$

→ Linear systems on $\mathbb{R}$

$$\dot{x} = \underline{a}\,x$$

Linearity cond

$$f(ax + bx') = a f(x) + b f(x')$$

$a > 0$

$a = 0$

line of fixed pts

$a < 0$

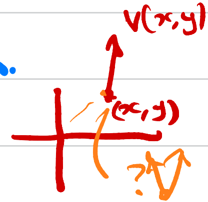
That's it!!

Wb.2

Chapter 4

Linear system in $\mathbb{R}^2$:

$$\left.\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}\right\} \quad a, b, c, d \in \mathbb{R}.$$



$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

vector field at the point (x,y) is $v(x,y) = (ax + by,\ cx + dy)$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

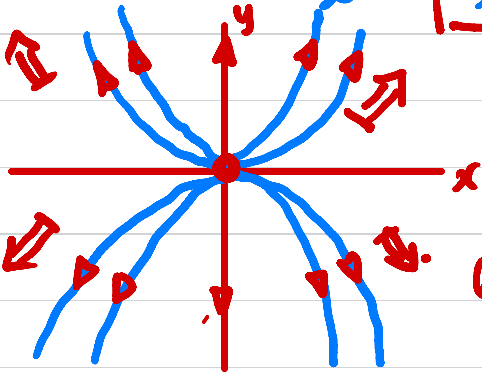
– consider this simple case (diagonal matrix)

w6.3

$$\dot{x} = ax \qquad (b=c=0) \qquad \dot{x} = x$$
$$\dot{y} = dy \qquad\qquad\qquad\qquad \dot{y} = 2y$$

FP $\dot{x}=0 \quad \begin{cases} x=0 \\ y=0 \end{cases}$
$\dot{y}=0$

elim dt $\quad \dfrac{\dot{y}}{\dot{x}} = \boxed{\dfrac{2y}{x} = \dfrac{dy}{dx}}$



$$\int 2\frac{dx}{x} = \int \frac{dy}{y}$$

$$2\ln x = \ln y + c \qquad c = -\ln(A)$$

$$\ln A + 2\ln(x) = \ln y$$

$$y = Ax^2$$

"unstable node"

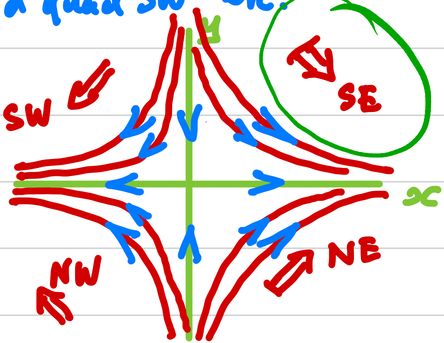
wb.4

$$\dot{x} = x$$
$$\dot{y} = -y$$

1st quad $x, y > 0$

$\therefore$ $\dot{x} > 0$ "SE"
$\dot{y} < 0$

2nd quad SW etc.



$$\frac{dy}{dx} = \frac{-y}{x}$$

$$\int \frac{dy}{y} + \frac{dx}{x} = 0$$

$$\ln(y) + \ln(x) = C$$

$$yx = C'$$, hyperbolas

phase portrait of a saddle.

LECTURES WEEK 6

W6.5

Linear Systems

$\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\qquad \dot{\underline{z}} = A\underline{z}$, $\qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad \underline{w} = \begin{bmatrix} u \\ v \end{bmatrix}$

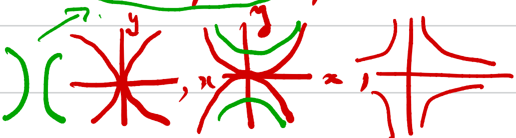
simple examples of A = diagonal matrix last time.

$\dot{x} = ax$, $\dot{y} = y$ for various values of a

simple to work out $dt = \dfrac{dx}{ax} = \dfrac{dy}{y} \Rightarrow x = cy^a$

$a > 1$

$a > 1$, $a < 1$, $a < 0$ etc.



node node saddle.

$\boxed{\dot{x} = y, \ \dot{y} = -w^2 x}$ tang to y-axis

$\boxed{H = y^2 + w^2 x^2}$

$0 < a < 1$ tang. to x-axis

Wb.6

How does H vary with time?

Consider

$$\dot{x} = y, \quad \dot{y} = -w^2 x,$$

$a = 0 \quad b = 1$

$c = -w^2 \quad d = 0$

$\varepsilon : \lambda^2 + w^2 = 0$

$\lambda = \pm i w$

dt?

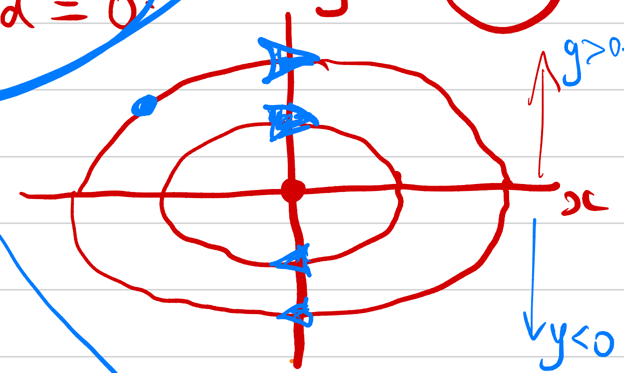
$$H = w^2 x^2 + y^2$$

$$H(t) = w^2 x(t)^2 + y^2(t)$$

$$\frac{dH}{dt}(t) = 2w^2 x \dot{x} + 2y\dot{y} =$$

$$= 2w^2 xy - 2y w^2 x \equiv 0$$

$$H(t) = \text{Constant}.$$



periodic orbits.

W6.7

Eigenvalues of A $=\{\lambda_1, \lambda_2\}$.

Eigenvectors of A $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$\lambda_1 \neq \lambda_2$

Note in linear systems {Solution curves / trajectories / orbits} lie along the

eigendirection (in general)

$\underline{v}_1 = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

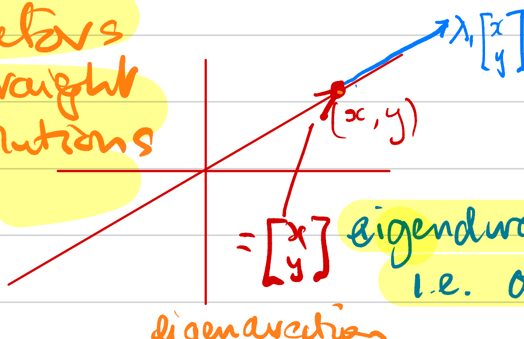
$A \begin{bmatrix} u \\ v \end{bmatrix} = \lambda_1 \begin{bmatrix} u \\ v \end{bmatrix}$

Suppose $A \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix}$, $v_1 =$ eigenvector of λ_1

$\lambda_1 u = \lambda_1 u$

$\lambda_2 v = \lambda_1 v$

eigenvectors give straight line solutions orbits.



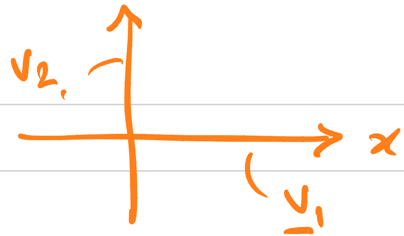
$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix}$

$\boxed{\lambda_1 \neq \lambda_2}$

eigendirection invariant under the flow i.e. orbits REMAIN on the eigendirection.

W6.8

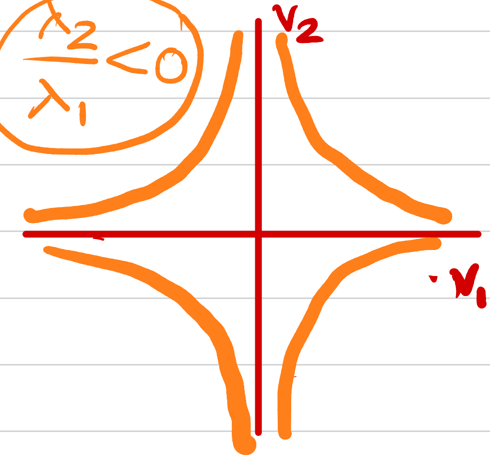
$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \checkmark$$



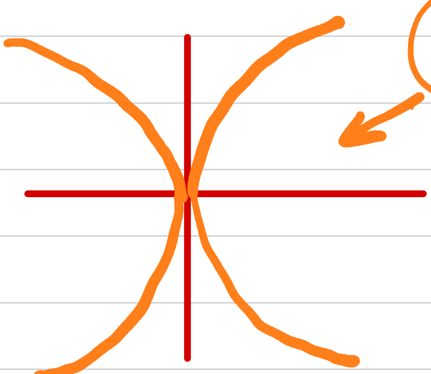
$$\frac{dx}{\lambda_1 x} = \frac{dy}{\lambda_2 y} \cancel{= dt}$$

$$y = C x^{\lambda_2 / \lambda_1}$$

$\frac{\lambda_2}{\lambda_1} < 0$



$\lambda_2 / \lambda_1 < 1$



$\frac{\lambda_2}{\lambda_1} > 1$

W6.9

Jordan forms

$\dot{\underline{z}} = A\underline{z}$, $\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\underline{w} = \begin{bmatrix} u \\ v \end{bmatrix}$ - new coordinates

Let $\underline{z} = P\underline{w}$ P is non-singular $\underline{P^{-1}}$ $\exists$

$\dot{\underline{z}} = A\underline{z}$, $\dot{\underline{z}} = P\dot{\underline{w}}$

$z = \begin{bmatrix} pu + qv \\ ru + sv \end{bmatrix}$

$P = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

constants

$P\dot{\underline{w}} = \underline{A}\underline{z} = \underline{A}\underline{P}\underline{w}$

$\dot{\underline{w}} = P^{-1}AP\underline{w}$

$= J\underline{w}$

Can I choose P such that $P^{-1}AP$ is "simpler" than A.

A, calculate eigenvalues λ_1, λ_2

$J = P^{-1}AP$ similar to A

$\mathcal{E}(J) = \mathcal{E}(A)$

λ_1, λ_2 real and $\lambda_1 \neq \lambda_2$

$\exists P$, $P^{-1}AP = \begin{bmatrix} \lambda_1 & \\ 0 & \lambda_2 \end{bmatrix}$ ✓

λ_1, λ_2 real and $\lambda_1 = \lambda_2$ $(= \lambda)$ but A is not diagonal

$\exists P$, $P^{-1}AP = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$

λ_1, λ_2 real $\lambda_1 = \lambda_2$ $(= \lambda)$ A is diagonal.

$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ — $\exists P = I$ s.t. $P^{-1}AP = J (= A)$

λ_1, λ_2 complex : $\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$ (A is real)

W6.11

$\exists P$ (real) s.t. $P^{-1}AP = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

p26, 27

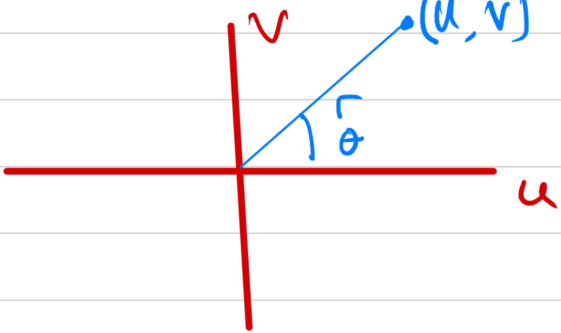
$\left[\exists Q \text{ s.t. } Q^{-1}AQ = \begin{bmatrix} \alpha+i\beta & 0 \\ 0 & \alpha-i\beta \end{bmatrix} \right]$ $\beta \neq 0$

Real $\rightarrow$ Complex

$\dot{u} = (\alpha+i\beta)u, \quad \dot{v} = (\alpha-i\beta)v$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$\underline{\dot{w}} = J\underline{w}$



Polar coordinates A-level.

$x = r\cos\theta$

$y = r\sin\theta$

p27 W6.12

$$\boxed{r^2 = x^2 + y^2 \quad , \quad r\dot{r} = x\dot{x} + y\dot{y}}$$

Polar coordinate

$$r\dot{r} = u\dot{u} + v\dot{v}$$
$$= u(\alpha u - \beta v) + v(\beta u + \alpha v)$$
$$= \alpha(u^2 + v^2) \qquad r\dot{r} = \alpha r^2$$
$$r\dot{r} = \alpha r^2 \Rightarrow \boxed{\dot{r} = \alpha r}$$

$\alpha > 0$, r incr with time
$\alpha < 0$ r decr with time

$r \geq 0$

$\dot{r} = \alpha r$

$\dot{\theta} = \beta$

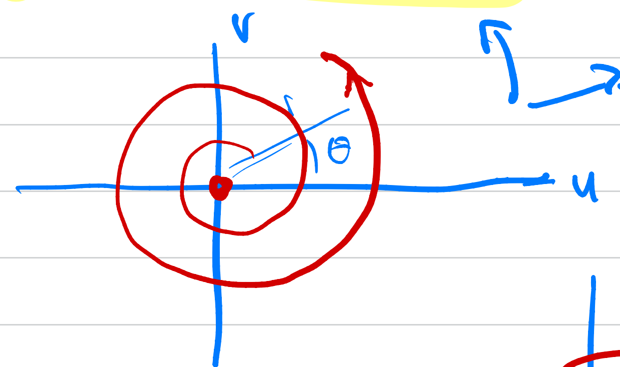
$$\boxed{r^2\dot{\theta} = x\dot{y} - y\dot{x}}$$

"uv coords" $\quad r^2\dot{\theta} = u\dot{v} - v\dot{u} = \beta(u^2+v^2) = \beta r^2$

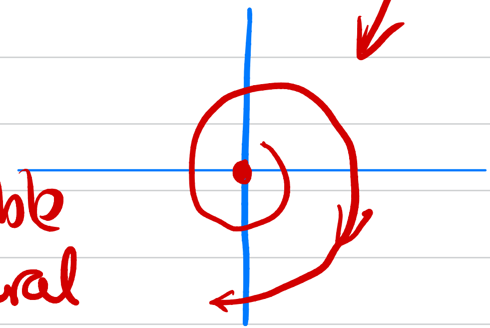
$$\boxed{\dot{\theta} = \beta}$$

General cases of complex eigenvalues

(i) $\alpha > 0$, $\beta > 0$ $\quad r = \alpha r \quad , \theta = \beta$



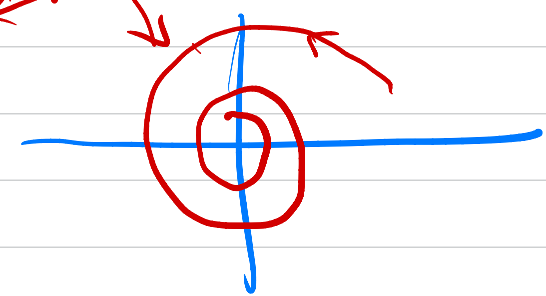
unstable spiral



(ii) $\alpha > 0$ $\quad \beta < 0$

(iii) $\alpha < 0$ $\quad \beta < 0$

stable spiral



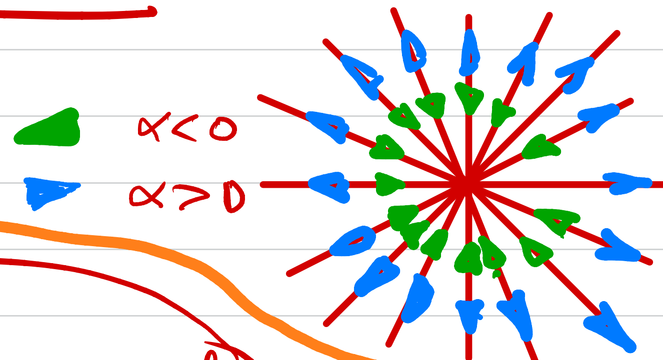
(iv) $\alpha < 0$ $\quad \beta > 0$

Special cases of complex eigenvalues

(v) $\alpha = 0$ $\quad \beta \neq 0$ → circles →

earlier example.
$\dot{x} = y$
$\dot{y} = -\omega^2 x$
$\beta = \omega$.
$\alpha = 0$.

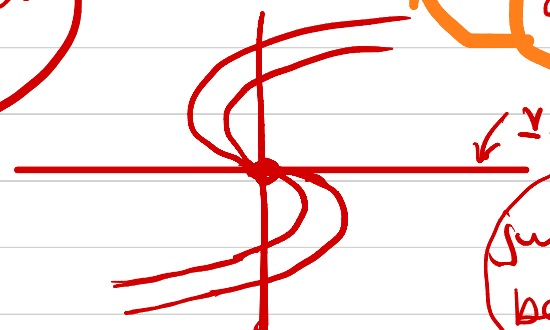
(vi) $\alpha \neq 0$ $\quad \beta = 0$ → radial lines



$\alpha > 0$
unstable
star node

$\alpha < 0$
stable star node

$\lambda_1 = \lambda_2$ non-diagonal
improper node
see Thursday tutorial

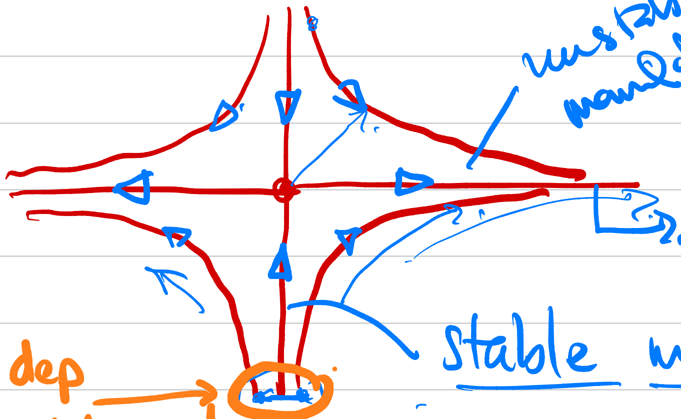


$\underline{v}_1$
just one eigendirection
because of repeated
eigenvalue

Note saddle is unstable

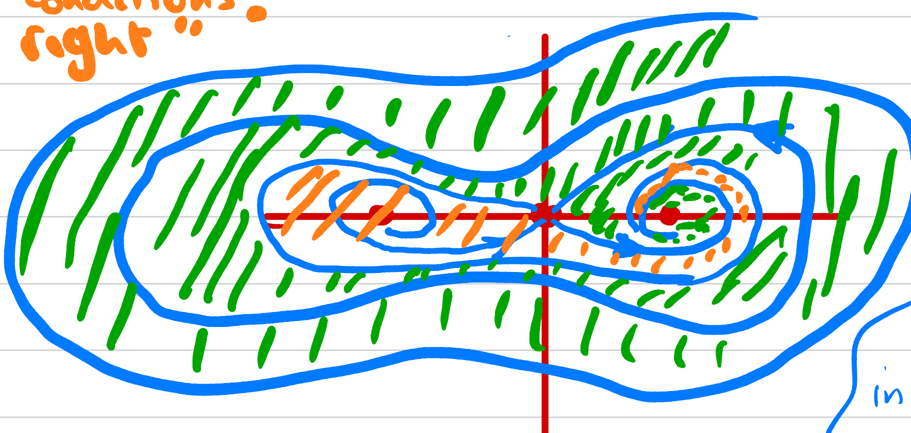
unstable manifold

$$\boxed{B(O) = y\text{-axis}}$$



stable manifold of the fixed pt

sensitive dep on initial conditions!
"left or right"



Nonlinear system – role of saddle manifolds is very important in dividing basins of attraction

W6.16

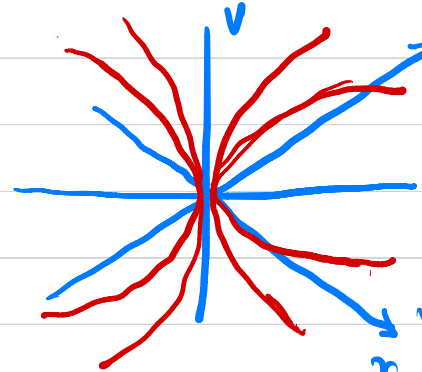
$\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \underline{w} = \begin{bmatrix} u \\ v \end{bmatrix}$

$\dot{\underline{z}} = A\underline{z} \Rightarrow \dot{\underline{w}} = \underbrace{P^{-1}AP}_{J}\underline{w}$

$\underline{z} = P\underline{w}$

Fig 15 p 30

$x = pu + qv$
$y = ru + sv$



y-axis $x=0$

x-axis $y=0$

$\dot{x}=0$ $pu+qv=0$

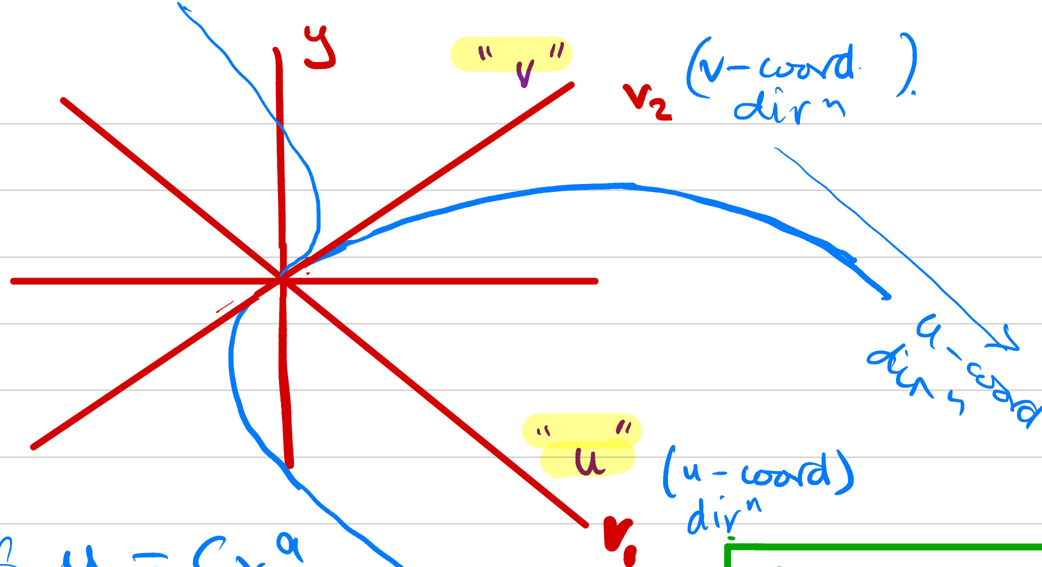
$\dot{y}=0$ $ru+sv=0$

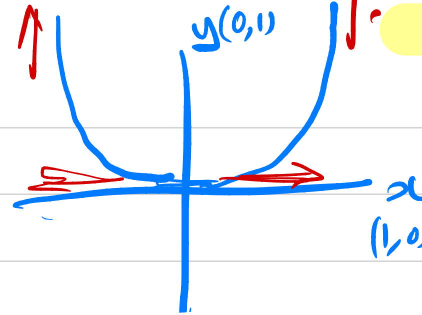
$\dot{\underline{w}} = \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 2u \\ v \end{bmatrix}$

$\frac{dv}{v} = \frac{du}{2u}$

suppose $u = cv^2$

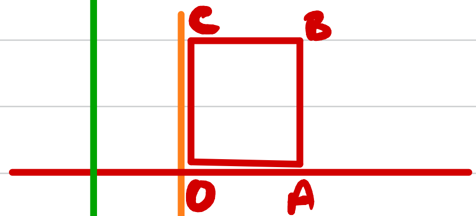
W6.17

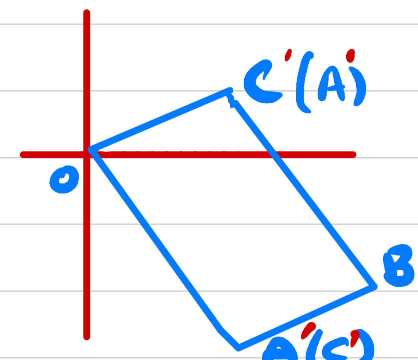




If $u = Cr^a$
$a > 1$
then curves tangent to v-axis at origin & turn in the u-dir^n at "∞"

Classic example of the linear transformation of a square.





$O \to O$
$A \to A'$ etc.

Revision of week 6.

WK8.1

$\dot{\underline{z}} = \underline{A}\,\underline{z}$

$\dot{\underline{w}} = J\,\underline{w}$

$\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix} \approx (x, y)$

$\underline{z} = \underline{P}\,\underline{w}$

$\underline{w} = \begin{bmatrix} u \\ v \end{bmatrix} \approx (u, v)$

$J = \underline{P}^{-1} A \underline{P}$

$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ λ_1, λ_2 real

$\lambda_1, \lambda_2 > 0$

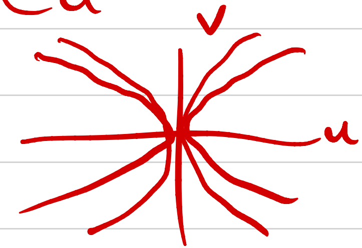
$\lambda_1, \lambda_2 < 0$

$\frac{du}{dt} = \lambda_1 u \,, \quad \frac{dv}{dt} = \lambda_2 v$

$\frac{du}{\lambda_1 u} = \frac{dv}{\lambda_2 v} \quad \Rightarrow \quad \ln v = \frac{\lambda_2}{\lambda_1} \ln(u) + C$

$$V = C u^{\lambda_2/\lambda_1}$$

$V = C u^{1/2} \Rightarrow u = C^2 v^2$

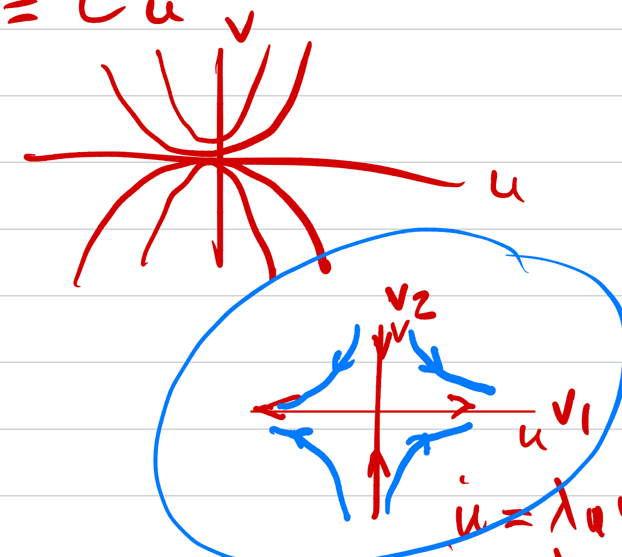


$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underline{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\underline{A}$ has λ_1, λ_2 eigvals

$\lambda_1 \cdot \lambda_2 < 0$

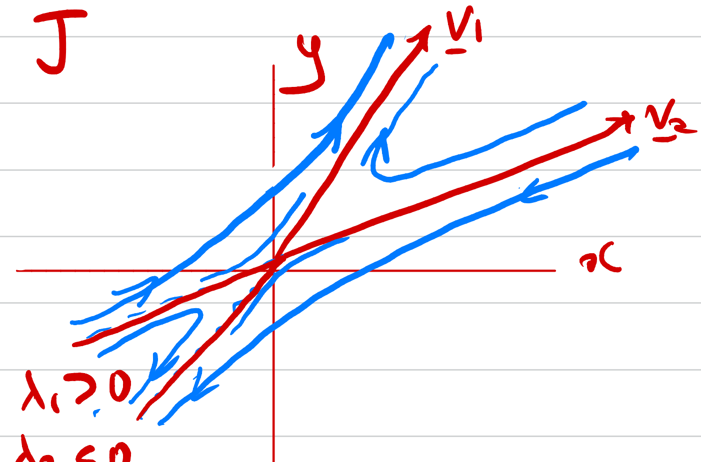
$V = C u^2$



$\dot{u} = \lambda_1 u$

$\dot{v} = \lambda_2 v$

J



$\lambda_1 > 0$

$\lambda_2 < 0$

Recap - linear systems and coordinates.

$$\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \underline{w} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \underline{z} = \underline{P}\underline{w}$$

$$\dot{\underline{z}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underline{A}\begin{bmatrix} x \\ y \end{bmatrix} = \underline{A}\,\underline{z} = \underline{A}\,\underline{P}\,\underline{w}$$

$$\| $$

$$\underline{P}\dot{\underline{w}} \quad \therefore \quad \underline{P}\dot{\underline{w}} = \underline{A}\,\underline{P}\,\underline{w}, \quad \dot{\underline{w}} = \underline{P}^{-1}\underline{A}\,\underline{P}\,\underline{w}, \quad \dot{\underline{w}} = \underline{J}\,\underline{w}$$

- we choose $\underline{P}$ to give a Jordan matrix $\underline{J}$
- remember $\mathcal{E}(\underline{J}) = \mathcal{E}(\underline{A})$ because $\underline{A}$ & $\underline{J}$ are "SIMILAR" matrices.

$$\boxed{\mathrm{Det}(\lambda\underline{I} - \underline{J})} = \mathrm{Det}(\lambda\underline{I} - \underline{P}^{-1}\underline{A}\,\underline{P}) = \mathrm{Det}(\underline{P}^{-1}(\lambda\underline{I} - \underline{A})\underline{P}) = \mathrm{Det}(\underline{P}^{-1})\,\mathrm{Det}(\lambda\underline{I} - \underline{A})\,\mathrm{Det}\,\underline{P}$$

$$= \boxed{\mathrm{Det}(\lambda\underline{I} - \underline{A})} \;\&\; \mathrm{Det}(P^{-1}) \cdot \mathrm{Det}(P) = \mathrm{Det}(PP^{-1}) = 1 \quad \blacksquare$$

Wk 8-3

Change of coordinates $\begin{bmatrix} x \\ y \end{bmatrix} \longleftrightarrow \begin{bmatrix} u \\ v \end{bmatrix}$

$\underline{z}$ $\quad\quad$ $\underline{w}$

Consider the case of $J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ λ_1, λ_2 real and $\lambda_1 \neq \lambda_2$

Eigenvectors: $\underline{v}_1$ for λ_1, $\underline{v}_2$ for λ_2

$A\underline{v}_1 = \lambda_1 \underline{v}_1$

$A\underline{v}_2 = \lambda_2 \underline{v}_2$

Choose $\underline{P} = [\underline{v}_1 \mid \underline{v}_2]$

$$\underline{A}\,\underline{P} = A\overbrace{[\underline{v}_1 \mid \underline{v}_2]}^{P} = [A\underline{v}_1 \mid A\underline{v}_2] = [\lambda_1\underline{v}_1 \mid \lambda_2\underline{v}_2] = \overbrace{[\underline{v}_1 \mid \underline{v}_2]}^{\underline{P}}\ \cdot\ \overbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}^{J}$$

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = J$$

WK8.4

$$\underline{z} = \underline{P}\,\underline{w} \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \underline{v}_1 , \underline{v}_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = u\,\underline{v}_1 + v\,\underline{v}_2$$

Coord.   Coords

$$x\,\underline{e}_1 + y\,\underline{e}_2$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = u \begin{bmatrix} \underline{v}_1 \end{bmatrix} + v \begin{bmatrix} \underline{v}_2 \end{bmatrix}$$

WK 8.5

Significance of change of coordinates.

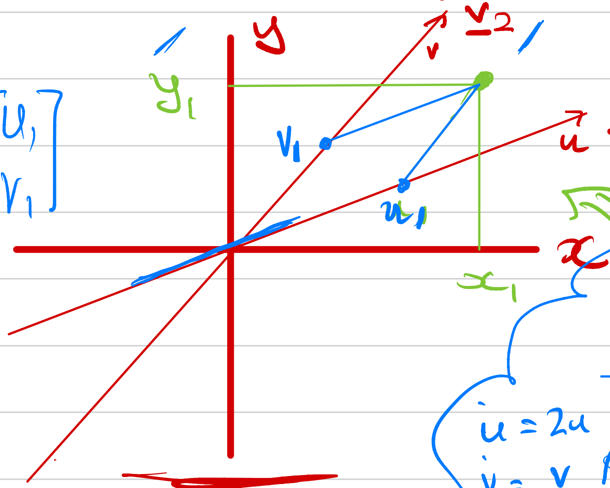
$$\underline{z} = \begin{bmatrix} x \\ y \end{bmatrix} = \underline{P} \begin{bmatrix} u \\ v \end{bmatrix}$$

Chosen $\underline{P} = [\underline{v}_1 \,\vdots\, \underline{v}_2]$ $\quad \begin{bmatrix} x \\ y \end{bmatrix} = [\underline{v}_1 \,\vdots\, \underline{v}_2] \begin{bmatrix} u \\ v \end{bmatrix} = u\,\underline{v}_1 + v\,\underline{v}_2$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \end{bmatrix} = u\,\underline{v}_1 + v\,\underline{v}_2$$

u, v coordinates

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = [\underline{v}_1 \,\vdots\, \underline{v}_2] \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$$



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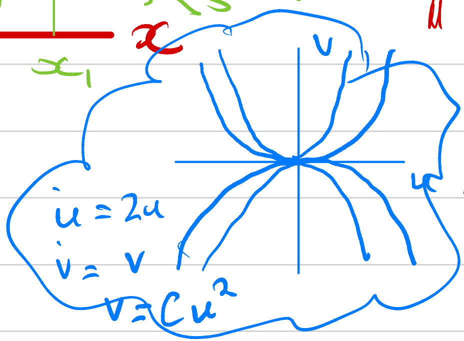
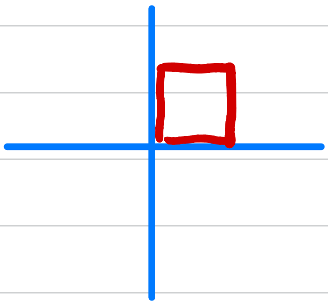


$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

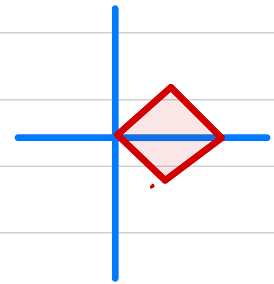
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

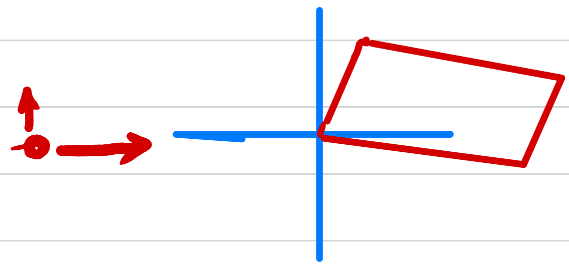
WK8.5(a)

Note effects of a linear transformation:

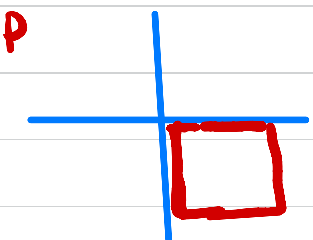


rotate





flip



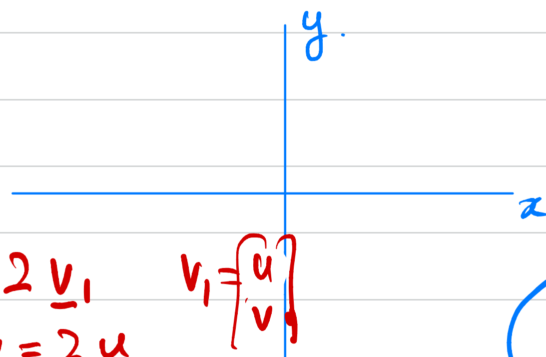
WK8.6

Example 4.4 page 32

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ ✓✓ , $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$.

"Jordan canonical form"

$\lambda_1 = \lambda_2 = 2$, $\underline{v}_1 = \underline{v}_2 = (1, 0)^T$



NW ?
NE.
SE
SW

$A\underline{v}_1 = 2\underline{v}_1$ $\quad v_1 = \begin{bmatrix} u \\ v \end{bmatrix}$

$2u + v = 2u$

$2v = 2v$

$v = 0 \quad \underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\dot{\underline{x}} = A\underline{x}$

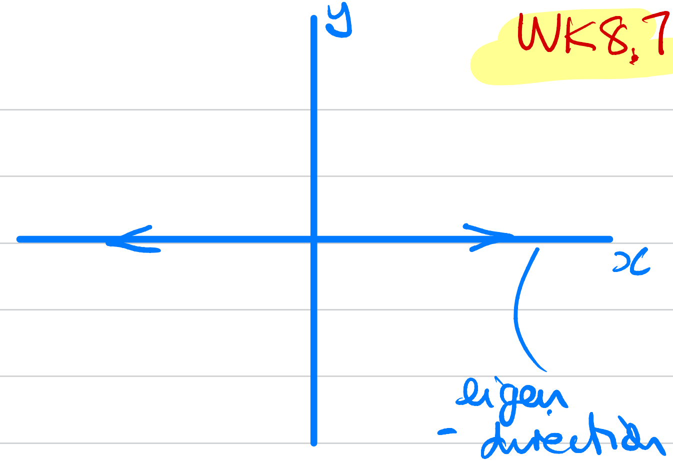
Node

Improper node

$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

WK8.7

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$



$$\lambda = \frac{Tr(A) \pm \sqrt{Tr(A)^2 - 4 \, Det \, A}}{2}$$

$$A\begin{bmatrix} x \\ 0 \end{bmatrix} = 2\begin{bmatrix} x \\ 0 \end{bmatrix}.$$

"invariant under the flow"

or "if on the eigendirection you stay on it."

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$Det \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix} = 0$$

$$(\lambda - a)(\lambda - d) - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{Tr(A)}\lambda + \underbrace{ad - bc}_{det(A)} = 0$$

WK8.8

$$\lambda = \frac{Tr(A) \pm \sqrt{Tr(A)^2 - 4\,Det(A)}}{2}$$

nodes / saddles

$Tr(A)^2 - 4\,Det(A) > 0$ — 2 real eigenvalues $\lambda_1 \neq \lambda_2$

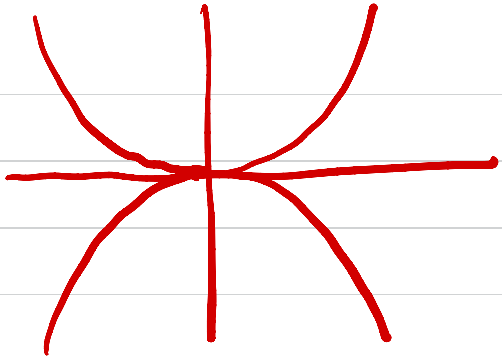
?? $Tr(A)^2 - 4\,Det(A) = 0$ — 2 real eigenvalues $\lambda_1 = \lambda_2$

$Tr(A)^2 - 4\,Det(A) < 0$ — 2 complex eigenvalues

spirals

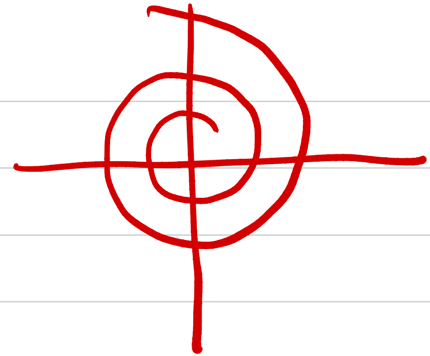
Fig 18 (p33) Eigenvalues fit into clear areas of the Tr-det plane. Eg. $\det(A) < 0 \Rightarrow$ saddles why because $\det(A) = \lambda_1 \cdot \lambda_2$ and $\lambda_1 \cdot \lambda_2 < 0$ only if λ_1, λ_2 are of opposite sign, etc.

WK8.9



nodes

$D > 0$



$D < 0$

Improper node.

"Half way between a node and a spiral"

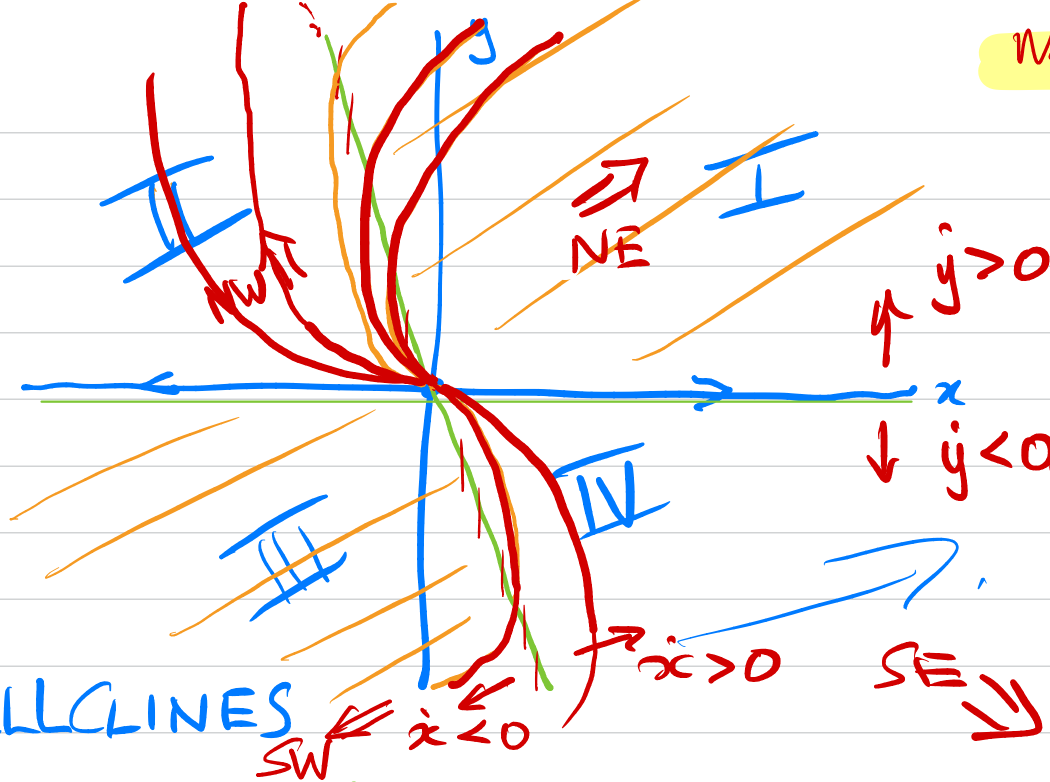
= IMPROPER NODE "

→ λ positive

← λ negative

WK8.10

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$



$$\dot{x} = 2x + y$$

$$\dot{y} = 2y$$

NULLCLINES

Consider

$\dot{x} = 0$ $\quad 2x + y = 0$ (no movement in x-dir^n $\therefore$ vertical dir^n only).

$\dot{y} = 0$ $\quad 2y = 0. \quad y = 0.$

moving horizontally

WK8.11

$A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$

$\mathrm{Det}(\lambda I - A) = 0$ — the eigenvalue equation

$\mathrm{Tr}(A) = 6$

$\lambda = \dfrac{6 \pm \sqrt{6^2 - 4.9}}{2}$

$\mathrm{Det}(A) = 9$

Reminder!

$\lambda = 3$ (rep).

unstable improper node

A repeated roots and non-diagonal

Nullclines are pts where $\dot{x} = 0$ or $\dot{y} = 0$

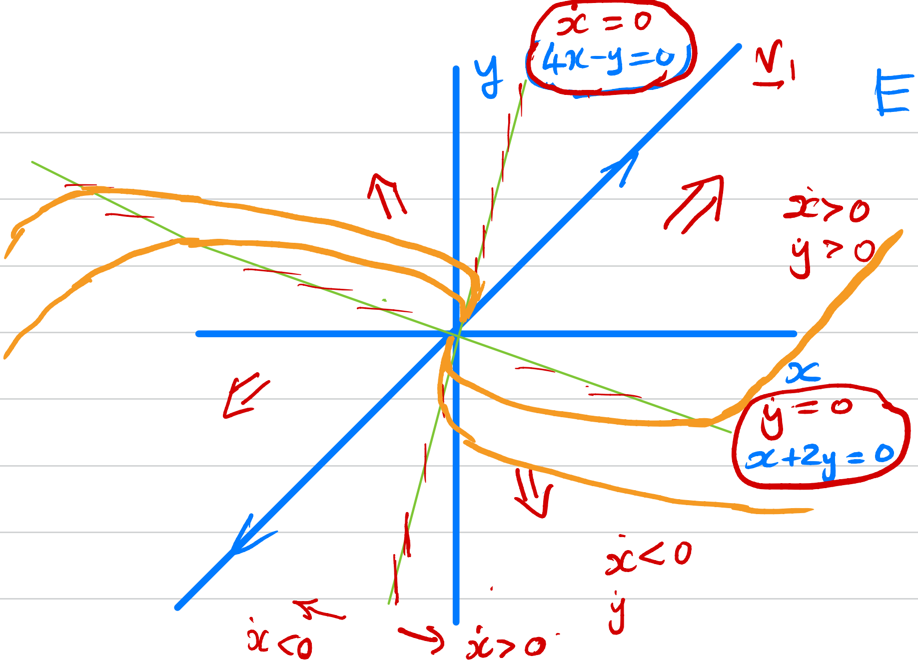
$\dot{x} = 0 \qquad 4x - y = 0 \qquad \updownarrow ?$

$\dot{y} = 0 \qquad x + 2y = 0 \qquad \leftrightarrow ?$ Horizontally . . . .

Any other isocline can be considered

e.g. $\dfrac{dy}{dx} = -1 \qquad \dfrac{4x - y}{x + 2y} = -1 \iff 4x - y = -x - 2y$

$\boxed{5x + y = 0}$



Eigenvector: $\begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$ Wk 8.12

$\Rightarrow 4x - y = 3x$ i.e. $x = y$

$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

linear saddle

p33 $\dot{x} = 0 \Rightarrow 4x - y = 0$

$\dot{y} = 0 \Rightarrow x + 2y = 0$

Defns at end of Chapter 4
"Attracting" is a loose definition
and will not be used in the
final exam.

WK8.13

There is much to come to terms with in today's class

But we do have practise in the last chapter of

- Locating fixed points
    - investigating eigenvalues and eigenvectors
        - nullclines
            - general direction of solution curves ("NE", etc)
                - sketching phase portraits.
                    - . . . . . .