Lecture 2

Lemma: $r_n = \frac{s_n}{t_n}$, $0 \le n \le N$

Proof: $r_0 = \frac{s_0}{t_0} = a$

Proof by induction. Let, $r_n = \frac{s_n}{t_n}$

$$= \frac{a_n s_{n-1} + s_{n-2}}{a_n t_{n-1} + t_{n-2}}.$$

Then $r_{n+1} = [a; a_1, \ldots, a_n, a_{n+1}]$

$$= [a; a_1, \ldots, a_n + \frac{1}{a_{n+1}}]$$

$$= \frac{\left(a_n + \frac{1}{a_{n+1}}\right) s_{n-1} + s_{n-2}}{\left(a_n + \frac{1}{a_{n+1}}\right) t_{n-1} + t_{n-2}}$$

$$= \frac{a_{n+1}\left(a_n s_{n-1} + s_{n-2}\right) + s_{n-1}}{a_{n+1}\left(a_n t_{n-1} + t_{n-2}\right) + t_{n-1}}$$

$$= \frac{a_{n+1} s_n + s_{n-1}}{a_{n+1} t_n + t_{n-1}}$$

$$= \frac{s_{n+1}}{t_{n+1}}$$ $\boxed{R}$

Example $\quad r = [3; 7, 15, 1]$

$= 3.141592$

$s_{-1} = 1 \qquad t_{-1} = 0$

$s_0 = 3 \qquad t_0 = 1$

$s_1 = a_i s_0 + s_{-1} = 22 \qquad t_1 = 7$

$s_2 = 333 \qquad t_2 = 106$

$s_3 = 555 \qquad t_3 = 113$

$r_1 = \frac{22}{7}, \quad r_2 = \frac{333}{106}, \quad r_3 = \frac{355}{113}$

$\quad \| \qquad\qquad \| \qquad\qquad \|$

$3.1428 \qquad 3.1415 \qquad 3.1416$

Theorem: 1) $s_n t_{n-1} - t_n s_{n-1} = (-1)^{n-1}$

2) $r_n - r_{n-1} = \frac{(-1)^{n-1}}{t_{n-1} t_n}$

3) $GCD(s_n, t_n) = 1$

Pf: 1) Induction: $s_0 t_{-1} - t_0 s_{-1}$

$= -1 = (-1)^{0-1}$

Let $s_{n-1} t_{n-2} - t_{n-1} s_{n-2} = (-1)^{n-2}$

Then check that $s_n t_{n-1} - t_n s_{n-1} = (-1)^{n-1}$

2) Immediate from def^n of r_n & part 1)

3) Immediate from part 1)

<u>Cor</u>: The convergents satisfy

$$r_0 < r_2 < r_4 < \dots < r_5 < r_3 < r_1$$

<u>Proof</u>: $r_{2i} - r_{2i+2} = r_{2i} - r_{2i+1} + r_{2i+1} - r_{2i+2}$

$$r_{n+2} - r_n = \frac{(-1)^{n+1}}{t_{n+2}\, t_{n+1}} + \frac{(-1)^n}{t_{n+1}\, t_n} = \frac{(-1)^n a_{n+2}}{t_{n+2}\, t_n}$$

$\Rightarrow$ $r_{n+2} > r_n$ if n even

$r_n > r_{n+2}$ if n odd

Further, $r_N - r_{N-1} = \dfrac{(-1)^N}{t_{N-1} \times t_N} < 0$

choosing $N = 2i + 2j$ even. Thus

$$r_{2i} < r_{2i+2j} < r_{2i+2j+1} < r_{2j-1}$$

$\forall i, j$. $\square$