Lecture 1

Continued Fraction

Recall the algorithm to find

$GCD(225, 157)$ $(= 1)$

$$\begin{aligned} &= GCD(68, 157) = GCD(68, 21) \\ &= GCD(5, 21) = GCD(1, 5) = 1. \end{aligned}$$

corresponding to Euclid's algorithm

$$\begin{aligned} 225 &= 157 \times 1 + 68 \\ 157 &= 68 \times 2 + 21 \\ 68 &= 21 \times 3 + 5 \\ 21 &= 5 \times 4 + 1 \end{aligned}$$

We may rewrite this as

$$\frac{21}{5} = 4 + \frac{1}{5}$$

$$\frac{68}{21} = 3 + \frac{5}{21} = 3 + \cfrac{1}{4 + \cfrac{1}{5}}$$

$$\frac{157}{68} = 2 + \frac{21}{68} = 2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}$$

$$\frac{225}{157} = 1 + \frac{68}{157} = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}}$$

This kind of expression is called <u>continued fraction</u>.

<u>Def</u>: <u>Finite Continued Fraction</u>: For $N \geq 1$, $a, a_1, \ldots, a_{N-1} \in \mathbb{Z}$ & $a_N \in \mathbb{R}$, we write FCF

$$[a; a_1, a_2, \ldots, a_N]$$ to denote

$$a + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots \, a_{N-1} + \frac{1}{a_N}}}}$$

If "$N = 0$" we mean $[a]$ by $a \in \mathbb{Z}$.

By definition: $[a; a_1, a_2]$

$$= a + \cfrac{1}{a_1 + \frac{1}{a_2}} = [a; a_1 + \tfrac{1}{a_2}]$$

Similarly,

$$[a; a_1, \ldots, a_N] = [a; a_1, \ldots, a_{N-1} + \tfrac{1}{a_N}]$$

$$= \left[a; a_1, \dots, a_{N-2} + \cfrac{1}{a_{N-1} + \cfrac{1}{a_N}}\right]$$

For example,

$$\frac{21}{5} = [4;5] = 4 + \frac{1}{5}$$

$$\frac{68}{21} = [3;4,5] = 3 + \cfrac{1}{4 + \cfrac{1}{5}}$$

$$\frac{225}{157} = [1;2,3,4,5]$$

<u>Prop</u>: Let $r = \frac{s}{t}$ be a rational number and $r > 1$ in lowest order, which means $GCD(s,t) = 1$. Then r can be written as $[a; a_1, \dots, a_N]$ with $a, a_1, \dots, a_N \in \mathbb{N}$, with $a_N > 1$.

<u>RMK</u>: If $t = 1$, then $r = [s;]$.

<u>RMK</u>: Conversely, for every $a, a_1, \dots, a_N$ with $a_N > 1$ the FCF $[a; a_1, \dots, a_N]$ defines a unique rational number > 1. [Convince yourself]

Proof

We run Euclid algorithm

$$s = at + t_1 \Rightarrow \frac{s}{t} = a + \frac{t_1}{t}$$

$$0 \le t_1 < t \Rightarrow \frac{t_1}{t} < 1$$

$$t = a_1 t_1 + t_2 \Rightarrow \frac{t}{t_1} = a_1 + \frac{t_2}{t_1}$$

$$\frac{s}{t} = a + \frac{1}{t/t_1} = a + \frac{1}{a_1 + \frac{t_2}{t_1}}$$

$$\vdots$$

$$r = \frac{s}{t} = a + \cfrac{1}{a_1 + \cfrac{1}{\ddots\, a_{N-1} + \cfrac{1}{a_N}}}$$

$$r = [a; a_1, \dots, a_N].$$

$\boxtimes$

A different (may be more direct) algorithm

Denote: $\rho = \frac{s}{t} = r$, $t_0 = t$

$$\rho_1 = \frac{t}{t_1}, \quad \rho_2 = \frac{t_1}{t_2}, \quad \dots, \quad \rho_j = \frac{t_{j-1}}{t_j}, \quad \dots$$

We would like to relate ρ_j with ρ_{j+1}.

$$\rho_j = \frac{t_{j-1}}{t_j} = a_j + \frac{t_{j+1}}{t_j} = a_j + \frac{1}{\rho_{j+1}}$$

$$\Rightarrow \quad \rho_{j+1} = \frac{1}{\rho_j - a_j}$$

This algorithm bypasses Euclid's algorithm.

<u>Def</u>: Floor function $\lfloor \ \rfloor : \mathbb{R} \to \mathbb{Z}$.

$\lfloor x \rfloor :=$ Largest integer N s.t. $N \leq x$.

<u>Ex!</u> $\lfloor 1.5 \rfloor = 1$, $\lfloor 3.999 \rfloor = 3$

$\lfloor \pi \rfloor = 3$, $\lfloor -5.6 \rfloor = -6$

$\lfloor n \rfloor = n \quad \forall n \in \mathbb{Z}$.

Note that, $\rho_j > 1 \Rightarrow \frac{1}{\rho_j} < 1$

Thus $\lfloor \rho_j \rfloor = a_j$

Algorithm : $a = \lfloor r \rfloor = \lfloor \rho \rfloor, \ \rho = \frac{s}{t}$

$$\rho_1 = \frac{1}{\rho - a} \rightsquigarrow a_1 = \lfloor \rho_1 \rfloor$$

$$\rho_2 = \frac{1}{\rho_1 - a_1} \rightsquigarrow a_2 = \lfloor \rho_2 \rfloor$$

$$\vdots$$

$$\rho_N = \frac{1}{\rho_{N-1} - a_{N-1}} \rightsquigarrow a_N = \lfloor \rho_N \rfloor$$

Then $r = [a; a_1, \ldots, a_N]$

Example : $r = \frac{87}{38}$

$$a = \left\lfloor \frac{87}{38} \right\rfloor = 2$$

$$\rho_1 = \frac{1}{\frac{87}{38} - 2} = \frac{38}{11} \rightsquigarrow a_1 = \lfloor \rho_1 \rfloor = 3$$

$$\rho_2 = \frac{1}{\frac{38}{11} - 2} = \frac{11}{5} \rightsquigarrow a_2 = \lfloor \rho_2 \rfloor = 2$$

$$\rho_3 = \frac{1}{\frac{11}{5} - 2} = 5 \rightsquigarrow a_3 = \lfloor \rho_3 \rfloor = 5$$

$$r = \frac{87}{38} = [2; 3, 2, 5] = 2 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{5}}}$$

Example : $x = -\frac{3}{5}$

$a = \lfloor -\frac{3}{5} \rfloor = -1$

$\rho_1 = \frac{1}{-\frac{3}{5}+1} = \frac{5}{2} \rightsquigarrow a_1 = \lfloor \rho_1 \rfloor = 2$

$\rho_2 = \frac{1}{\frac{5}{2}-2} = 2 \rightsquigarrow a_2 = \lfloor \rho_2 \rfloor = 2$

So, $-\frac{3}{5} = [-1;\ 2, 2]$

Example : $x = \frac{3}{5}$

$a = [\frac{3}{5}] = 0$

$\rho_1 = \frac{1}{\frac{3}{5}-0} = \frac{5}{3} \rightsquigarrow a_1 = 1$

$\rho_2 = \frac{1}{\frac{5}{3}-1} = \frac{3}{2} \rightsquigarrow a_2 = 1$

$\rho_3 = \frac{1}{\frac{3}{2}-1} = 2 \rightsquigarrow a_3 = 2$

$\frac{3}{5} = [0;\ 1, 1, 2]$

RMK: If the last term of continued fraction is >1 then it is unique.

Theorem: Let $r \in \mathbb{Q}$ s.t. $r > 0$

$$r = [a; a_1, \ldots, a_m] = [b; b_1, \ldots, b_n]$$

with $a_m, b_n > 1$ and $a, \ldots, a_m, b_1, \ldots, b_n \in \mathbb{N}$. Then $k = l$ and $a_j = b_j \ \forall j$

Proof: Induction on

Let $m = 0 \Rightarrow r = a$

$$= b + \frac{1}{[b_1; b_2, \ldots, b_n]}$$

$\Rightarrow \quad 0 \leq a - b < 1$

But $a, b \in \mathbb{N} \Rightarrow a = b = r$

Let the assertion hold for $\cdot 1,$

i.e., $$r = a + \frac{1}{[a_1; a_2, \ldots a_m]} = b + \frac{1}{[b_1; b_2, \ldots, b_n]}$$

$$= a + a' = b + b'$$

where $a, b \in \mathbb{N}$ & $\underbrace{0 \leq a', b' < 1}_{\text{follows from last proof}}$

$\lfloor r \rfloor = a = b$

$\Rightarrow a' = b'$

From induction hypothesis we have $m-1 = n-1$ & $a_j = b_j$.

<u>Def</u>: Let $a \in \mathbb{Z}$ and $a_1, \ldots, a_N \geq 1$. Let $r_n := [a; a_1, \ldots, a_n]$ $0 \leq n \leq N$. We call $r_0, r_1, \ldots, r_N$ to be <u>convergents</u> of $r := [a; a_1, \ldots, a_N]$

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Let $s_{-2} = 0$, $s_{-1} = 1$, $s_0 = a$

$$s_n = a_n s_{n-1} + s_{n-2}$$

$t_{-2} = 1$, $t_{-1} = 0$, $t_0 = 1$

$$t_n = a_n t_{n-1} + t_{n-2}$$

<u>Exercise</u>: Check that s_n & t_n are increasing sequences.