① Selected solutions to problem set 6

2. The solution is

$$u(x,t) = \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right]$$

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} 1 \, ds$$

$$+ \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} (-r)\, dr\, ds$$

$$= \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right] + t$$

$$- \frac{1}{2c}\int_0^t \frac{(x+ct-cs)^2 - (x-ct+cs)^2}{2}\, ds$$

$$= \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right] + t$$

$$+ \left.\frac{(x+ct-cs)^3}{12c^2}\right|_0^t + \left.\frac{(x-ct+cs)^3}{12c^2}\right|_0^t$$

$$= \frac{1}{2}\left[e^{-(x-ct)^2} + e^{-(x+ct)^2}\right] + t$$

$$+ \frac{x^3}{6c^2} + \frac{(x+ct)^3 + (x-ct)^3}{12c^2}$$

3. The solution is

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} 1 \, dr \, ds$$

$$= \frac{1}{2c}\int_0^t \left[(x+ct-cs) - (x-ct+cs)\right] ds$$

$$= \frac{1}{2c}\int_0^t (2ct - 2cs)\, ds$$

$$= \frac{1}{2c} \cdot \left[2cts - cs^2\right]\Big|_0^t$$

$$= \frac{1}{2c}\left[2ct^2 - ct^2\right]$$

$$= \frac{t^2}{2}$$

③ 5. The general solutions are.

$$U(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nct) + b_n \sin(nx)\sin(nct)$$

with

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cdot \sin(nx)\,dx$$

$$= \begin{cases} 0 & , n \neq 1 \\ \frac{2}{\pi}\int_0^{\pi} \frac{1-\cos(2x)}{2}\,dx & , n=1 \end{cases}$$

$$= \begin{cases} 0 & , n \neq 1 \\ 1 & , n=1 \end{cases}$$

So the solution is

$$u(x,t) = \sin x \cos(ct)$$

6. (1) For $x > 0$,

$F(x) = f(x)$

and becase $-x < 0$,

$F(-x) = f(-(-x)) = f(x) = F(x)$

So F is even.

G is even by the same reason.

The derivative of an even function is always odd.

(2) The solution is

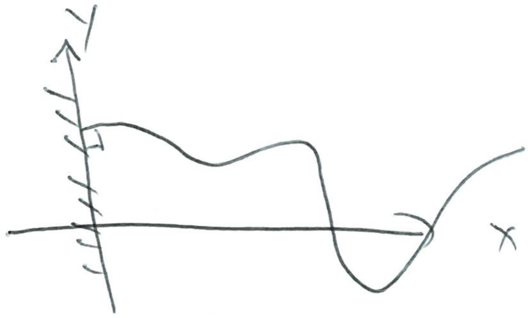
$$V(x,t) = \frac{1}{2}[F(x+ct) + F(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,ds$$

(3) $V_x(x,t) = \frac{1}{2}[F'(x+ct) + F'(x-ct)] + \frac{1}{2c}G(x+ct) - \frac{1}{2c}G(x-ct)$

(4) $V_x(0,t) = \frac{1}{2}[F'(ct) + F'(-ct)] + \frac{1}{2c}[G(ct) - G(-ct)] = 0$

because F' is odd and G is even.

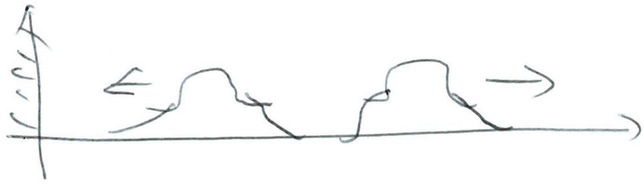
(5)



The string is allowed to move up and down, but always orthognoal to the y axis at the left end.

For an intial bump 

It first separate into 2 bumps



~~After~~ ~~at~~
After some time the left bump hit the "wall" (y-axis) and is reflected.
then both bumps are propagating to right

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8. • The Energy is

$$E[u](t) = \int_{-\infty}^{\infty} \left[\frac{1}{2} u_t^2 + \frac{1}{2} a^2 u_x^2 \right] dx$$

Multiply both sides of equation by u_t, we get

$$u_t \cdot u_{tt} - a^2 u_t u_{xx} - c u_t^2 = 0$$

integrate get

$$\int_{-\infty}^{\infty} u_t \cdot u_{tt} dx \quad \int_{-\infty}^{\infty} a^2 u_t u_{xx} dx - c^2 \int_{-\infty}^{\infty} u_t^2 dx = 0$$

$$\int_{-\infty}^{\infty} \frac{u_t^2}{2} \cdot dx$$

$$\int_{-\infty}^{\infty} - \int_{-\infty}^{\infty} a^2 u_t u_{xx} - c^2 \int_{-\infty}^{\infty} u_t^2 dx = 0$$

By integration by part., we get

$$\int_{-\infty}^{\infty} \frac{(u_t^2)_t}{2} dx - a^2 u_t \cdot u_x \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} a^2 u_{xt} u_x dx - c^2 \int_{-\infty}^{\infty} u_t^2 dx = 0$$

thus

(5) $\frac{d}{dt}\int_{-\infty}^{\infty}\frac{(u_t)^2}{2}dx - 0 + \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\frac{a^2(u_x)^2}{2}dx$

$$- c\int_{-\infty}^{\infty} u_t^2 = 0.$$

Namely

$$\frac{d}{dt}\int_{-\infty}^{\infty}\left(\frac{u_t^2}{2} + \frac{a^2 u_x^2}{2}\right)dx = c\int_{-\infty}^{\infty} u_t^2$$

$$\leq 0$$

So the energy is non-increasing

because $c < 0$.

- Suppose u_1, u_2 are two solutions to
$$\begin{cases} u_{tt} - a^2 u_{xx} - c u_t = \cancel{f}(x), & c<0 \\ u(x,0) = f(x), & u_t(x,0) = g(x) \end{cases}$$

then $u = u_1 - u_2$ is a solution to

$$\begin{cases} u_{tt} - a^2 u_{xx} - c u_t = 0, & c < 0 \\ u(x,0) = 0, & u_t(x,0) = 0 \end{cases}$$

(b) We see ~~that~~ from the first part ~~of~~ the question that

$$\frac{d}{dt} E[u](t) = 0,$$

But $E[u](0) = \int 0^2 + 0^2 \, dx = 0$

So $E[u](t) \equiv 0$ for all t.

Namely $u_t \equiv 0$, $u_x \equiv 0$

and so $u \equiv 0$,

we must have $u_1 = u_2$

and the solution to

$$\begin{cases} u_{tt} - a^2 u_{xx} - c u_t = \psi(x), & c < 0 \\ u(x,0) = f(x), \ u_t(x,0) = g(x) \end{cases}$$

is unique.