Selected solutions to problem set 5.

1. The eigenvalue probem is

$$\begin{cases} X'' = -\lambda X \\ X'(0) = 0 \\ X(L) = 0 \end{cases}$$

The general solution is

$$\cancel{X_n(x)=} \quad X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$

$$X'(x) = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$$

$$X'(0) = C_2\sqrt{\lambda} = 0$$

implies $C_2 = 0$

$C_1 \neq 0$,

using $X(L) = 0$, we get

$$\sqrt{\lambda}\sin(\sqrt{\lambda}\cdot L) = 0$$

$$\sqrt{\lambda}L = \frac{\pi}{2} + n\pi.$$

So $\lambda_n = \dfrac{(\frac{1}{2}+n)^2\pi^2}{L^2}$

and $X_n(x) = \cos\left[(\frac{1}{2}+n)\frac{\pi x}{L}\right]$

Selected solutions to problem set 5. ①

2. We first find solutions using separation of variables

For solutions of the form

$$u(x,t) = X(x)T(t)$$

$$X\ddot{T} = c^2 X'' \cdot T$$

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X} = -\lambda.$$

$$\begin{cases} \ddot{T} = -c^2\lambda T \\ X'' = -\lambda X \end{cases}$$

Now we solve X using the condition

$$u_x(0,t) = u_x(\pi,t) = 0,$$

we get
$$\begin{cases} X'' = -\lambda X \\ X'(0) = 0 \\ X'(\pi) = 0. \end{cases} \quad (*1)$$

claim: $\lambda \geq 0$. (This is similar to the argument in lecture notes)

Proof of claim: Multiply both sides by X, get

$$XX'' = -\lambda X^2.$$

$$XX'' + \lambda X^2 = 0$$

So $0 = \int_0^\pi (XX'' + \lambda X^2)\, dx \underset{\substack{\uparrow \\ \text{integration by parts}}}{=} XX'\Big|_0^\pi - \int_0^\pi (X')^2 dx + \lambda \int_0^\pi X^2 dx$

Since $XX'\big|_0^\pi = \cancel{0}\ 0$ by the condition, we have

$$0 = -\int_0^\pi (X')^2 dx + \lambda \int_0^\pi X^2 dx$$

$$\int_0^\pi (X')^2 dx = \lambda \cdot \int_0^\pi X^2 dx,$$

$\Rightarrow \quad \lambda \geq 0.$

so (*1) has solutions (using the method for solving 2nd order ODEs) given by

$$X(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$$

Differentiate get, $X'(x) = -\sqrt{\lambda} C_1 \sin(\sqrt{\lambda} x) + C_2\sqrt{\lambda} \cos(\sqrt{\lambda} x)$.

By the boundary conditions, we get

$$X'(0) = C_2\sqrt{\lambda} \cos(0) = 0$$

so $C_2 = 0$,

and $C_1 \neq 0$.

$$X'(\pi) = -\sqrt{\lambda} \sin(\sqrt{\lambda}\pi) = 0$$

This implies $\sqrt{\lambda}\pi = n\pi$

$\lambda_n = n^2$ are eigenvalues

and $X_n(x) = \cos(nx)$, $n = 0, 1, \cdots$

To solve T, we have ③

$$\ddot{T} = -\lambda_n c^2 T = -n^2 c^2 T,$$

~~So~~ when $n=0$, $T_0(t) = d_1 + d_2 t$

when $n \geq 1$, $T_n(t) = d_1 \cos(nct) + d_2 \sin(nct)$

So
$$\begin{cases} u_n(x,t) = a_n \cos(nx)\cos(nct) + b_n \cos(nx)\sin(nct), & n \neq 0 \\ u_0(x,t) = a_0 + b_0 t, & n = 0 \end{cases}$$

And $u(x,t) = (a_0 + b_0 t) + \sum_{n=1}^{\infty} [a_n \cos(nx)\cos(nct) + b_n \cos(nx)\sin(nct)]$

Now, use $u(x,0) = 0$, we get

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = 0,$$

so $a_0 = 0$, $a_n = 0$

So $u(x,t) = b_0 t + \sum_{n=1}^{\infty} b_n \cos(nx) + \sin(nct)$

~~Use $u_t(x,0) = 0$, we get~~

So ~~since~~ $u_t(x,t) = b_0 + \sum_{n=1}^{\infty} (nc) b_n \cos(nx)\cos(nct)$

Thus $u_t(x,0) = b_0 + \sum_{n=1}^{\infty} (nc) b_n \cos nx$,

Using $u_t(x,0)$ ~~$=0$, we then get~~
$= \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$.

we then get $b_0 = \frac{1}{2}$, $b_2 = \frac{1}{4c}$, $b_n = 0$ for $n \neq 0, 2$

So the solution is

$$u(x,t) = \frac{1}{2}t + \frac{1}{4c}\cos(2x)\sin(2ct).$$

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3. The Fourier coeffients are

$$a_0 = \frac{1}{2L} \int_{-L}^{L} |x| \, dx$$

$$= \frac{1}{L} \int_0^L x \, dx$$

$$= \frac{1}{L} \cdot \frac{L^2}{2} = \frac{L}{2}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} |x| \cdot \cos\left(\frac{\pi n x}{L}\right) dx$$

$$= \frac{2}{L} \int_0^L x \cos \frac{\pi n x}{L} \, dx$$

$$= \frac{2}{L} \cdot \frac{x \cdot L}{\pi n} \sin \frac{\pi n x}{L} \Big|_0^L - \frac{2}{L} \int_0^L \frac{L}{\pi n} \sin \frac{\pi n x}{L} dx$$

$$= \frac{-2}{L} \cdot \frac{-L^2}{(\pi n)^2} \cos\left(\frac{\pi n x}{L}\right) \Big|_0^L$$

$$= \frac{2L}{(\pi n)^2} \left[\cos(n\pi) - \cos(0) \right]$$

$$= \frac{2L}{(\pi n)^2} \left[(-1)^n - 1 \right].$$

and thus $\hat{f}(x) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{(\pi n)^2} \left[(-1)^n - 1 \right] \cos\left(\frac{\pi n x}{L}\right)$