Periodic points of higher period

Prop If $f: \mathbb{R} \to \mathbb{R}$ is an order-preserving diffeomorphism then any periodic point for f must be a fixed point.
(i.e. There are no points of least period k if $k > 1$).

Proof Let $x_0 \in \mathbb{R}$. As usual, let $x_n = f^n(x_0)$.
If x_0 is not a fixed point then <u>either</u> $x_1 > x_0$ <u>or</u> $x_1 < x_0$.

If ~~$x_1 < x_0$~~ $x_0 < x_1$ then $f(x_0) < f(x_1)$ since f is order-preserving, i.e. $x_1 < x_2$.
But then $f(x_1) < f(x_2)$, i.e. $x_2 < x_3$.
So $x_0 < x_1 < x_2 < x_3 < \ldots$
So in particular, $x_k \neq x_0$ for all $k \geq 1$, therefore x_0 is not periodic.

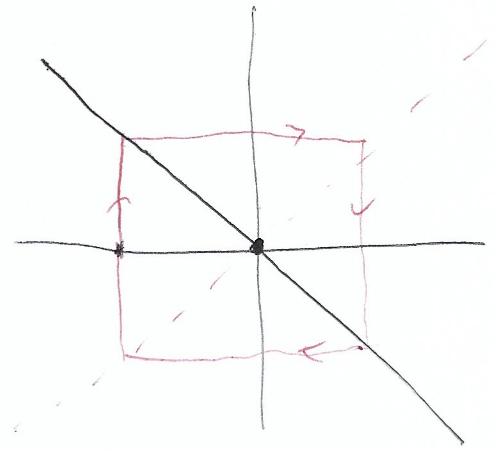
Similarly, if $x_0 > x_1$ then the order-preserving property, applied repeatedly, implies

$$x_0 > x_1 > x_2 > x_3 > \dots$$

and again we see x_0 is <u>not</u> periodic. $\square$

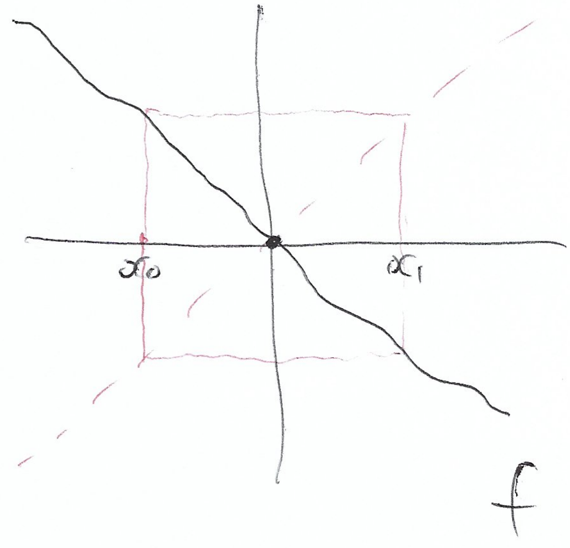
What about periodic points for order-reversing diffeomorphisms?

<u>Example</u> $f(x) = -x$



Here, 0 is the unique fixed point, but <u>every other</u> point has least period 2

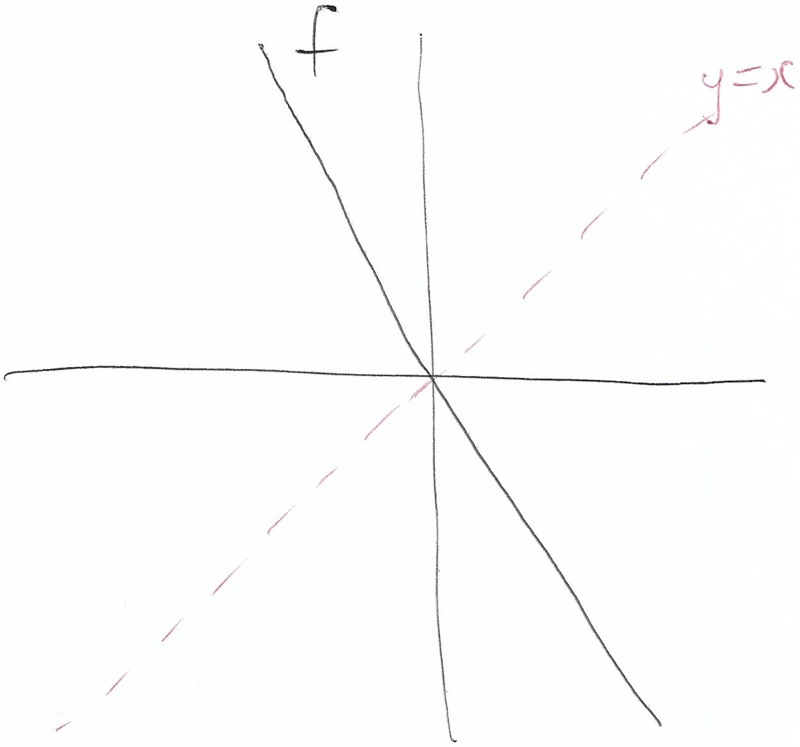
<u>Example</u> Graphical illustration of another order-reversing diffeo with a 2-cycle $\{x_0, x_1\}$



Example

$f(x) = -2x$

Here there are no points of least period 2



As we see with these examples, ~~there~~ for order-reversing diffeomorphisms there could be points of least period 2, but there need not be.

Rather like the situation for fixed pts of order-preserving differs, we will not prove any general result about least period 2 points for o-r differs.

<u>Proposition</u> If $f:\mathbb{R}\to\mathbb{R}$ is an order-reversing diffeomorphism then there are no points of least period strictly larger than 2.

<u>Proof</u> Since f is a diffeomorphism, so is $f^2 = f\circ f$. Notice that f^2 is order-<u>preserving</u> because

$$(f^2)'(x) = \underbrace{f'(f(x))}_{<0} \cdot \underbrace{f'(x)}_{<0} ,$$

which is >0 (since $f'<0$, and the product of 2 ~~positive~~ negative values is ~~negative~~ positive).

So by the previous Proposition, f^2 does not have any periodic points of least period > 1.

So f itself does not have any

periodic points of least period $2m$ for $m > 1$, i.e. f has no least period k points for <u>even</u> numbers $k > 2$.

To address the case where k is odd (and ≥ 3) we note that for such k, the map f^k is an order-reversing diffeomorphism. But order-reversing diffeos have <u>precisely</u> one fixed point, and of course the unique ~~pe~~ fixed point p for f is also a fixed point for f^k therefore p is the unique fixed point for f^k. So f^k has no other fixed points. In other words, there are no period-k points for f, except for the fixed pt p. i.e. There are no points of <u>least</u> period k for f. $\square$

Summarising the situation for all diffeomorphisms $f: \mathbb{R} \to \mathbb{R}$:

	Fixed Points	Least period 2	Least period > 2
f order-preserving	Arbitrary number	None	None
f order-reversing	Exactly one	Arbitrary number	None

Dynamics of continuous maps f

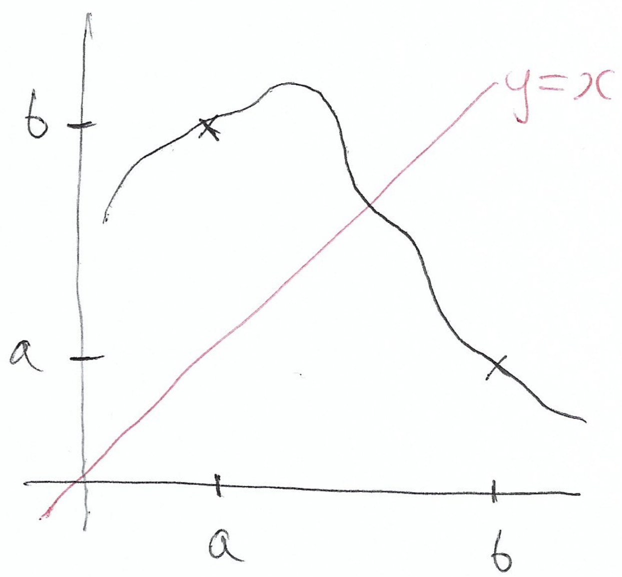
– fixed points and periodic points

Proposition Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If f has an orbit of least period 2 then it has a fixed point.

Proof Let $\{a, b\}$ be a 2-cycle for f, with $a < b$.

Then $f(a) = b$, and $f(b) = a$.

Let $g(x) := f(x) - x$, so that a zero of g is a fixed point of f.



Clearly f is continuous on $[a,b]$, since it is continuous on $\mathbb{R}$.

Also $g(a) = f(a) - a$

$= b - a > 0$

and $g(b) = f(b) - b$

$= a - b < 0$.

By the Intermediate Value Theorem (applied to the continuous function g) there exists $c \in (a, b)$ such that $g(c) = 0$,

i.e. such that $f(c) = c$.

So f has a fixed point c. $\square$

<u>Remark</u> This is an example of "forcing", i.e. presence of a periodic orbit of some period <u>forces</u> the presence of an orbit of another period.

Perhaps the most famous result of this type is the "Period-3 implies chaos" theorem, which more precisely states:

<u>Theorem</u> If $f: \mathbb{R} \to \mathbb{R}$ is continuous and has an orbit of least period 3, then it has periodic orbits of <u>all</u> other least periods n, for $n \in \mathbb{N}$.

The previous 'period-3 implies chaos' theorem is a consequence of:

Sharkovskii's Theorem

Rearrange $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ with the following ordering (called the Sharkovskii ordering):

$$1 \lhd 2 \lhd 4 \lhd 8 = 2^3 \lhd 2^4 \lhd 2^5 \lhd \dots \lhd 2^n \lhd \dots$$

$$\vdots$$

$$\dots \lhd 2^k(2n+1) \lhd \dots \lhd 2^k \times 7 \lhd 2^k \times 5 \lhd 2^k \times 3$$

$$\vdots$$

$$\dots \lhd 2^2(2n+1) \lhd \dots \lhd 2^2 \times 7 \lhd 2^2 \times 5 \lhd 2^2 \times 3$$

$$\dots \lhd 2(2n+1) \lhd \dots \lhd 2 \times 7 \lhd 2 \times 5 = 10 \lhd 2 \times 3$$

$$\dots \lhd (2n+1) \lhd \dots \lhd 11 \lhd 9 \lhd 7 \lhd 5 \lhd 3$$

Let $I = (a, b) \subseteq \mathbb{R}$ (we allow $I = \mathbb{R} = (-\infty, \infty)$)

Suppose $f : I \to I$ is continuous, and has an m-cycle (i.e. a periodic orbit of least period m).

Then f has an n-cycle for all those natural numbers n for which $n \lhd m$.

<u>Example</u> Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous and has an orbit of least period 8.

Then by Sharkovskii's Theorem, f has a fixed point, and a 2-cycle, and a 4-cycle.

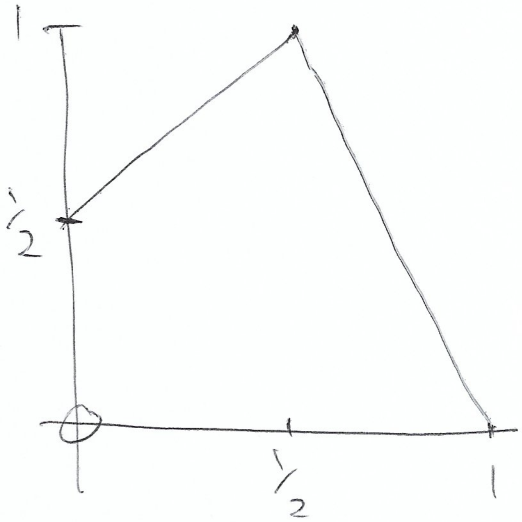
<u>Example</u> Suppose $f: \mathbb{R} \to \mathbb{R}$ is continuous and has a 10-cycle.

Then, by Sharkovskii's Theorem, f has n-cycles for all even numbers n except $n=6$.

Example Let $f: [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x \leq \frac{1}{2} \\ 2 - 2x & \text{if } x > \frac{1}{2} \end{cases}$$

Fixed point at $x = 2/3$



Period-2 orbit:

Assume $x \in [0, \frac{1}{2}]$. Solve $f^2(x) = x$.

i.e. $f(f(x)) = x$

i.e. $f(x + \frac{1}{2}) = x$

i.e. $2 - 2(x + \frac{1}{2}) = x$

i.e. $2 - 2x - 1 = x$

i.e. $1 = 3x$ i.e. $x = 1/3$

Therefore $f(1/3) = 1/3 + 1/2 = 5/6$ is also a period-2 orbit. So $\{1/3, 5/6\}$ is a 2-cycle.

There is also a 3-cycle, namely

$$\{0, \tfrac{1}{2}, 1\}.$$

Therefore, by Sharkovskii's Theorem, this f has n-cycles for all $n \in \mathbb{N}$.

Logistic family of maps, period-doubling, and universal constants

Definition The logistic family is the family of functions f_μ defined by

$$f_\mu(x) = \mu x(1-x)$$

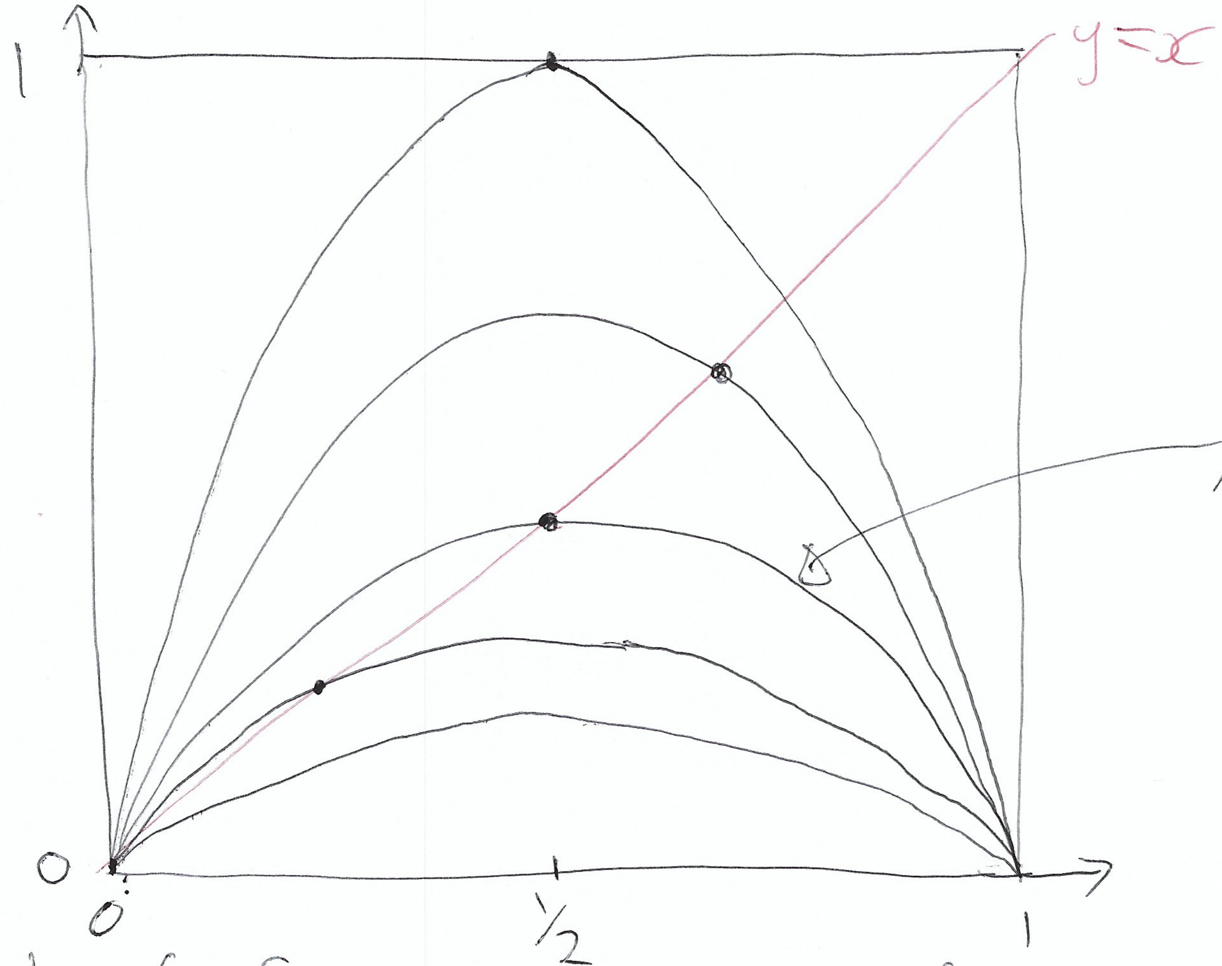
where $\mu > 0$ is a parameter.

As usual, we will study the dynamical system given by f_μ,

i.e. $x_{n+1} = \mu x_n(1-x_n)$

- First studied by R. May (1976) to model insect population dynamics.
- Today we treat this system as a model which has a transition to chaos

By convention we shall assume $x \in [0,1]$ (i.e. we consider $f_\mu : [0,1] \to [0,1]$), and $\mu \in [0,4]$.



Graphs of f_μ for various values of μ

<u>Derivative</u>: $f_\mu'(x) = \mu - 2\mu x$ (since f_μ is $f_\mu(x) = \mu x - \mu x^2$)

Critical point (maximum) at $x = \frac{1}{2}$ for <u>all</u> values of μ.

The maximum <u>value</u> of f_μ is then $f_\mu(\frac{1}{2}) = \frac{\mu}{4}$.

<u>Note</u>: $f_\mu(x) = f_\mu(1-x)$ for all x, so the graph of f_μ is symmetric about the point $\frac{1}{2}$.

<u>Fixed points</u> $\quad f_\mu(x) = x$

i.e. $\mu x(1-x) = x$

i.e. $0 = \mu x^2 + (1-\mu)x$

i.e. $x = 0$ or $0 = \mu x + 1 - \mu$

i.e. $x = \dfrac{\mu-1}{\mu} = 1 - 1/\mu$

So $x=0$ and $x = \dfrac{\mu-1}{\mu}$ are the fixed points of f_μ, though note that if $\mu<1$ then the 'fixed point' $\dfrac{\mu-1}{\mu}$ is <u>negative</u>, therefore outside of $[0,1]$, so we do not consider it (since we are thinking of f_μ as a map $[0,1] \circlearrowleft$).

When are these fixed points attracting?

We first calculate $|f_\mu'(x)|$:

$$|f_\mu'(x)| = |\mu - 2\mu x|$$

$$= \begin{cases} \mu & \text{if } x = 0 \\ |2-\mu| & \text{if } x = \frac{\mu-1}{\mu} \end{cases}$$

So for each fixed point we can (using a previous Theorem) say:

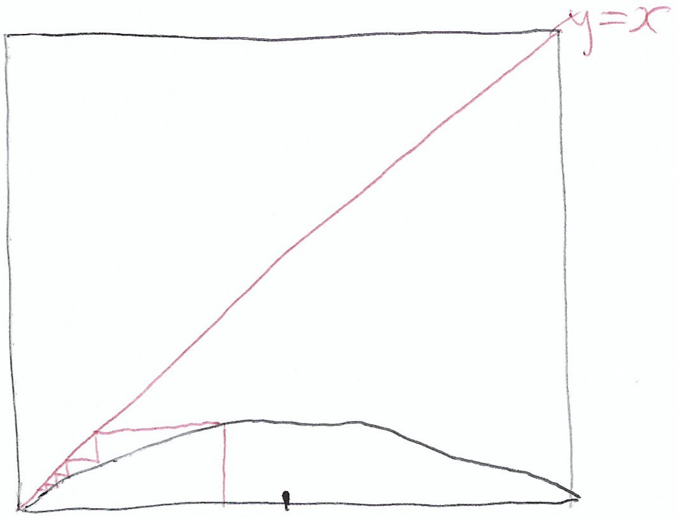
- The fixed point 0 is attracting if $0 \le \mu < 1$, and is repelling if $1 < \mu \le 4$.
- The fixed point at $\frac{\mu-1}{\mu}$ is attracting if $|2-\mu| < 1$,

  i.e. if $1 < \mu < 3$,

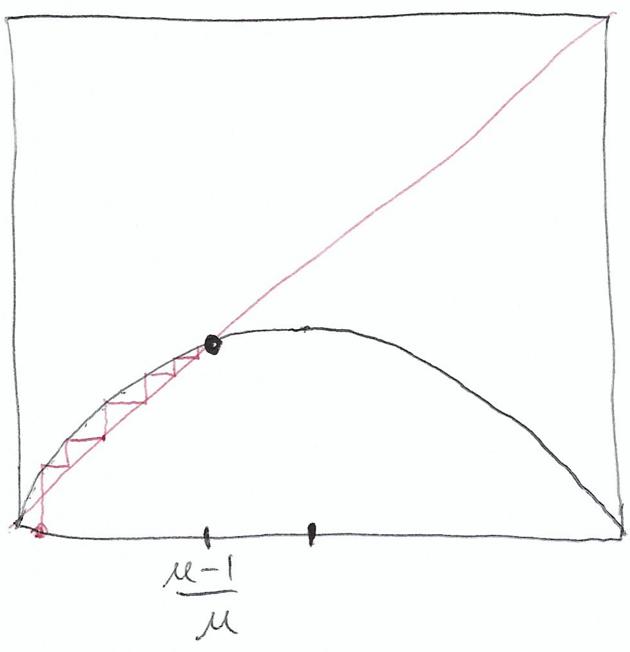
  and is repelling if $3 < \mu \le 4$

<u>Remark</u> So $\mu = 1$ is a key transition parameter, since the fixed point 0 stops being attracting, and the 'new' fixed point $\frac{\mu-1}{\mu}$ is 'born'.

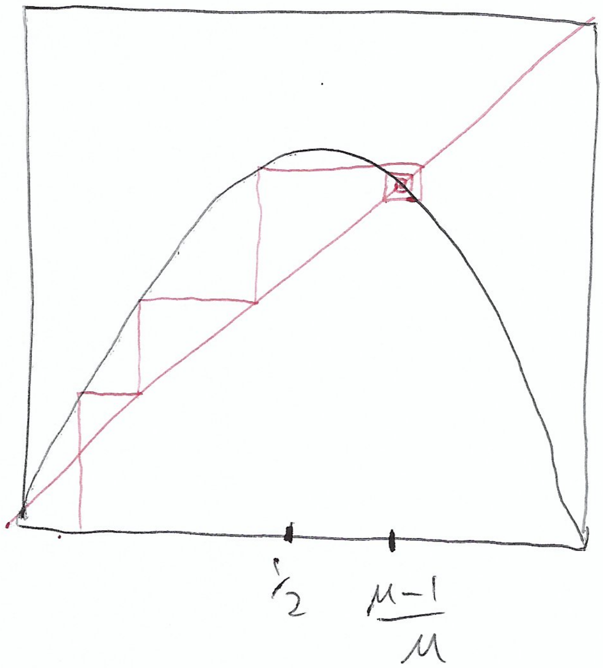
<u>Graph of f_μ</u> (for $0 \leq \mu < 1$)



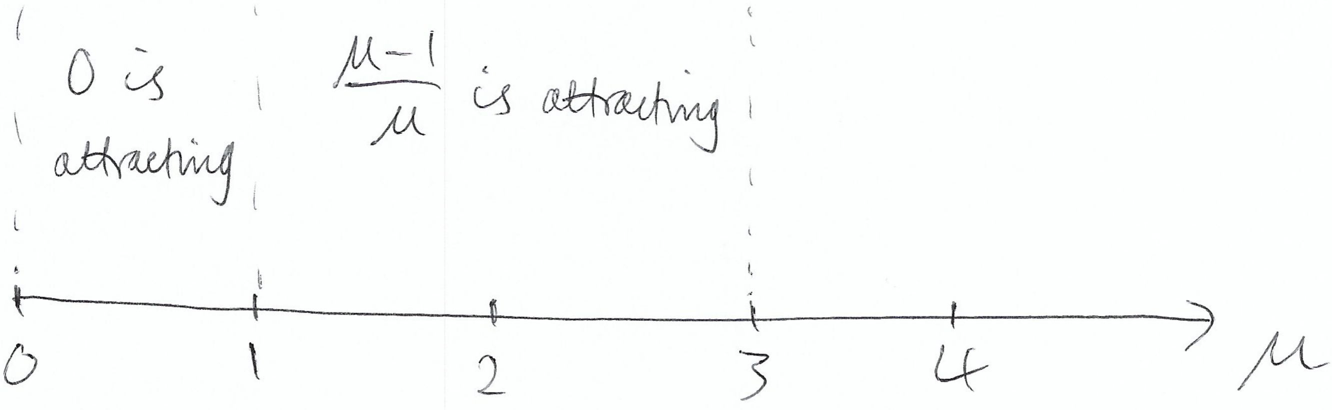
<u>Graph of f_μ</u> (for $1 < \mu < 2$)



Graph of f_μ (for $2 < \mu < 3$)



So far we can summarise as follows:



For $\mu > 3$, both fixed points 0 and $\frac{\mu-1}{\mu}$ are repelling, but what is happening dynamically? We will see that $\mu = 3$ marks the 'birth' of a period-2 orbit, and for those $\mu > 3$ which 'are not too much larger than 3' this period-2 orbit is <u>attracting</u>