Week 4

II. We are given that

(1) $$f_{X,Y}(x,y) = g(x)\,h(y)$$

We shall show that $f_{X,Y}(x,y) = f_X(x)\,f_Y(y)$

and thus prove the statement.

Let us find $f_X(x)$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{\infty} g(x)h(y)\,dy$$

$$= g(x)\underbrace{\int_{-\infty}^{\infty} h(y)\,dy}_{\shortparallel\ H} = H\cdot g(x),$$

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = H \times \underbrace{\int_{-\infty}^{\infty} g(x)\,dx}_{\shortparallel\ G} = HG$$

We see that $H \times G = 1$

Next $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{-\infty}^{\infty} g(x)h(y)\,dx$$

$$= h(y)\int_{-\infty}^{\infty} g(x)\,dx = G \times h(y)$$

Hence

$$f_X(x)\,f_Y(y) = H\,g(x)\,G\,h(y) = g(x)h(y) = f_{XY}(x,y)$$

We proved (1) and thus also the theorem. $\blacksquare$

<u>Example</u>. $f_{X,Y}(x,y) = \begin{cases} 2x & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$

Are r.v.'s X, Y independent?

<u>Solution</u>. $\underbrace{f_{X,Y}(x,y) = h(x).g(y)}$, where

$h(x) = \begin{cases} x & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$g(y) = \begin{cases} 2 & \text{if } 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$

Then $h(x)g(y) = 2x$ if $0<x<1,\ 0<y<1$,

and $h(x)g(y) = 0$ in all other cases

<u>Remark</u>. We could define $h(x) = 2x$ for $0<x<1$. Then $g(y) = 1$ for $0<y<1$.

<u>Example</u>. $f_{X,Y}(x,y) = \begin{cases} 2e^{-x-y} & \text{if } y>x>0 \\ 0 & \text{otherwise.} \end{cases}$

Are X and Y independent?

<u>Solution 1</u>. For $x>0$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_x^{\infty} 2e^{-x-y}\,dy = 2e^{-2x}$$

For $x \le 0$, $f_X(x) = 0$.

Next, $f_Y(y) = 0$ if $y \le 0$. For $y>0$ we have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^y 2e^{-x-y}\,dx$$

$$= 2e^{-y}\left(-e^{-x}\Big|_{x=0}^{y}\right) = 2e^{-y}\left(e^{0} - e^{-y}\right)$$

$$= 2e^{-y}(1-e^{-y})$$

If, say, $y > x > 0$ then

$$f_X(x)\,f_Y(y) = 4e^{-x-y}(1-e^{-y}) \neq f_{X,Y}(x,y) = 0$$

Hence X, Y are not independent. $\blacksquare$

Remark. The right way to present $f_X(x)$ and $f_Y(y)$ is

$$f_X(x) = \begin{cases} 2e^{-2x}, & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

$$f_Y(y) = \begin{cases} 2e^{-y}(1-e^{-y}) & \text{for } y > 0 \\ 0 & \text{if } y \le 0 \end{cases}$$

Solution 2. Note that $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy > 0$ if $x > 0$ since $f_{X,Y}(x,y) \le 0$ for $y > x$.

Similarly, $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx > 0$ if $y > 0$ since $f_{X,Y}(x,y) > 0$ for $x \in (0, y)$.

Hence $f_X(x)\,f_Y(y) > 0$ for any (x,y), $x > 0, y > 0$

We know that $f_{X,Y}(x,y) = 0$ if $x \ge y > 0$

So $f_{X,Y}(x,y) \neq f_X(x)\,f_Y(y) \Rightarrow$ not independent.

More properties of independent r.v.'s

I. Expectation of a product of $\varphi(X) \times \Psi(Y)$

<u>Exercise</u>. Prove that if X, Y are independent r.v.'s, then

$$\mathbb{E}(\varphi(X)\Psi(Y)) = \mathbb{E}(\varphi(X)) \times \mathbb{E}(\Psi(Y)).$$

<u>Proof</u>. By the Theorem saying that

$$\mathbb{E}(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

we have:

$$\mathbb{E}(\varphi(X)\Psi(Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x) g(y) f_X(x) f_Y(y)\, dx\, dy$$

(because $f_{XY} = f_X f_Y$)

$$= \int_{-\infty}^{\infty} \varphi(x) f_X(x)\, dx \times \int_{-\infty}^{\infty} \Psi(y) f_Y(y)\, dy$$

$$= \mathbb{E}(\varphi(X)) \times \mathbb{E}(\Psi(Y)). \quad \blacksquare$$

<u>Exercise</u> Deduce that if X, Y are independent, then $\mathbb{E}(X^k Y^m) = \mathbb{E}(X^k) \times \mathbb{E}(Y^m)$ for any k, m.

<u>Exercise</u>. Prove that $Cov(X,Y) = 0$ if X, Y are independent.

II. Sums of independent r.v.'s

Def-n. If $f(x)$, $g(x)$ are two continuous functions, then

$$(f*g)(z) \stackrel{def}{=} \int_{-\infty}^{\infty} f(z-y)\, g(y)\, dy$$

and $f*g$ is called the convolution of f and g.

Theorem. If X, Y are independent r.v.'s with pdf functions $f_X(x)$ and $f_Y(y)$, then their sum $Z = X+Y$ is a r.v. with pdf $f_Z(z)$ given by the convolution of f_X and f_Y:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y)\, f_Y(y)\, dy.$$

Proof. Left as exercise. Not examinable.

Example. $X \sim Exp(\lambda)$, $Y \sim Exp(\lambda)$. Find the pdf of $Z = X+Y$.

Solution. $f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda} & \text{if } y > 0 \\ 0 & \text{if } y \le 0 . \end{cases}$$

So

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy$$

$$= \int_0^{\infty} f_X(z-y) f_Y(y)\, dy .$$

If $z \le 0$ then $f_X(z-y) = 0$ for $y \ge 0$ and so $f_Z(z) = 0$.

If $z > 0$ then $f_X(z-y) > 0$ only if $y < z$.

So $f_Z(z) = \int_0^z f_X(z-y) f_Y(y)\, dy$

$$= \int_0^z \lambda^2 e^{-\lambda(z-y)} e^{-\lambda y}\, dy$$

$$= \lambda^2 \int_0^z e^{-\lambda z}\, dy = \lambda^2 e^{-\lambda z} z$$

So $f_Z(z) = \begin{cases} \lambda^2 z\, e^{-\lambda z} & \text{if } z > 0 \\ 0 & \text{if } z \le 0 . \end{cases}$

$\square$

Distribution of functions of r.v.'s.

Statement of the problem.

Given X with pdf $f_X(x)$, find the pdf of $Y=g(X)$, where $g: \mathbb{R} \to \mathbb{R}$.

Example. $f_X(x) = \begin{cases} 3x^2 & \text{if } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

$Y = 2X+3$. Find $f_Y(y)$.

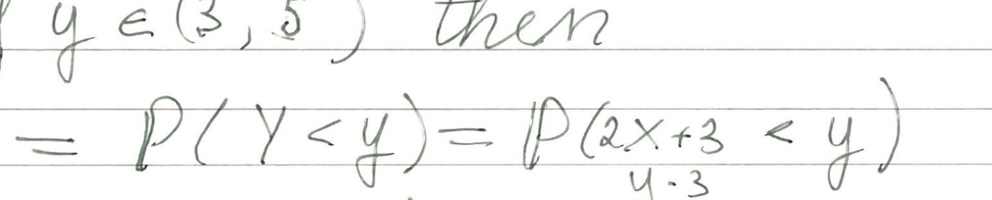
Solution. We first find $F_Y(y)$ - the cdf of Y using $f_Y(y) = \dfrac{dF_Y(y)}{dy}$.

By def-n $F_Y(y) = \mathbb{P}(Y \le y)$.

Note that the range of Y is $(3,5)$. Hence

$F_Y(y) = 0$ if $y \le 3$ $(\mathbb{P}(Y \le 3) = 0$.

$F_Y(y) = 1$ if $y \ge 5$.



If $y \in (3,5)$ then

$F_Y(y) = P(Y < y) = \mathbb{P}(2X+3 < y)$

$= \mathbb{P}\left(X < \dfrac{y-3}{2}\right) = \displaystyle\int_0^{\frac{y-3}{2}} f_X(x)\,dx$

$$= \int_0^{(y-3)/2} 3x^2\,dx = x^3 \Big|_{x=0}^{\frac{y-3}{2}} = \left(\frac{y-3}{2}\right)^3 .$$

So $f_Y(y) = 0$ if $y \notin (3,5)$,

$$f_Y(y) = \frac{d}{dy}\left(\frac{y-3}{2}\right)^3 = \frac{3}{8}(y-3)^2 .$$

$$f_Y(y) = \begin{cases} \frac{3}{8}(y-3)^2 & \text{if } y \in (3,5) \\ 0 & \text{otherwise.} \end{cases} \quad \boxtimes$$

The general method.

$X_1, X_2, \dots, X_n$ are r.v.'s with joint pdf $f_{\overline{X}}(\overline{x})$. $\overline{X} \overset{\text{def}}{=} (X_1, X_2, \dots, X_n)$,
$\overline{x} \overset{\text{def}}{=} (x_1, x_2, \dots, x_n)$

$$Y = g(\overline{X}), \quad g : \mathbb{R}^n \to \mathbb{R}$$

<u>Question</u>. What is the pdf of Y, - or rather how do we compute $f_Y(y)$?

<u>Answer</u>.

1. Find the region in $\mathbb{R}^n$ where $g(\overline{x}) \le y$. Let it be $\mathcal{D}$
2. Compute $F_Y(y) = \mathbb{P}(Y \le y) = \int_{\mathcal{D}} \dots \int f_{\overline{X}}(\overline{x})\,dx_1 \dots dx_n$
3. $f_Y(y) = \dfrac{d F_Y(y)}{dy}$.

<u>Example</u>. $Z \sim N(0,1)$. Find the pdf of $Y = Z^2$.

<u>Remarks</u> 1) $Z^2 \sim \chi^2(1)$

2) The distribution of $Z_1^2 + \dots + Z_n^2 \sim \chi^2(n)$, wher $Z_i \sim N(0,1)$ and $Z_1, \dots, Z_n$ are independent r.v.'s.

<u>Solution</u>. Var $g(y) = y^2$, $Y \in [0, \infty)$

Hence $f_Y(y) = 0$ if $y \leq 0$.

If $y > 0$, then we compute $F_Y(y)$.

$$F_Y(y) = P(Y \leq y) = P(Z^2 \leq y)$$

$$= P(-\sqrt{y} \leq Z \leq \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_Z(x)\,dx$$

$$Z \sim N(0,1) \iff f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$f_Y(y) = \frac{dF(y)}{d(y)} = f_Z(\sqrt{y}) \cdot (\sqrt{y})' - f_Z(-\sqrt{y})(-\sqrt{y})'$$

$$= \frac{1}{2} y^{-\frac{1}{2}} \left(f_Z(\sqrt{y}) + f_Z(-\sqrt{y}) \right)$$

Since $f_Z(\sqrt{y}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} = f_Z(-\sqrt{y})$, we get $f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}$. $\square$

Remark If $F(y) = \int_{a(y)}^{b(y)} f(x)\,dx$, then

$$F'(y) = f(b(y))\,b'(y) - f(a(y))\,a'(y)$$

Finding $f_Y(y)$ by the method of direct transformation of one r.v.

Theorem. Let X be a r.v. with pdf $f_X(x)$, $x \in [a,b]$. Let $g : [a,b] \to \mathbb{R}$ be a continuously differentiable function and $Y = g(X)$. Suppose also that g is monotonic, and g^{-1} exists. $g^{-1} : I \to \mathbb{R}$, where $I = g([a,b])$.

Then

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y))\left|\frac{d}{dy} g^{-1}(y)\right|, & \text{if } y \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise. Prove this theorem.

Example. $f_X(x) = \begin{cases} \frac{\theta}{x^{\theta+1}}, & \text{if } x > 1 \\ 0, & \text{if } x \le 0 \end{cases}$

$Y = g(X) = \ln X$, $\quad \theta > 0$.

Find $f_Y(y)$.

<u>Solution</u>. $Y = \ln X \Leftrightarrow X = e^{Y} (= g^{-1}(Y))$

$(g'(y) = x)$. Since $X > 1$, $Y > 0$.

So $f_Y(y) = 0$ if $y \le 0$.

If $y > 0$, then $(e^y)' = e^y$ and

$$f_Y(y) = \frac{\theta}{(e^y)^{\theta+1}} \times e^y = \frac{\theta e^y}{e^{\theta y + y}} = \theta e^{-\theta y}.$$

Thus $$f_Y(y) = \begin{cases} \theta e^{-\theta y} & \text{if } y > 0 \\ 0 & \text{if } y \le 0. \end{cases}$$ ◻

Hence $Y \sim Exp(\theta)$.

<u>Remark</u> X in this example has the <u>Pareto</u> distribution with parameter $\theta > 0$.

Week 4 Tutorial.

1) $\mathbb{E}(XY|Y=y) = \mathbb{E}(Xy|Y=y) = y\,\mathbb{E}(X|Y=y)$

Since X, Y are independen, $\mathbb{P}(X=x|Y=y)$

$= \mathbb{P}(X=x)$. Hence $\mathbb{E}(X|Y=y) = \sum_x x\,\mathbb{P}(X=x|Y=y)$

$= \sum_x x\,\mathbb{P}(X=x) = \mathbb{E}(X)$. So

$\mathbb{E}(X|Y=y) = \mathbb{E}(X)$ and

$\mathbb{E}(XY|Y=y) = y\,\mathbb{E}(X)$.

2) $f_{XY}(x,y) = \begin{cases} e^{-x-y} & \text{if } x>0,\ y>0 \\ 0 & \text{otherwise.} \end{cases}$

$h(x) = \begin{cases} e^{-x} & \text{if } x>0 \\ 0 & \text{if } x \le 0 \end{cases}$

$g(y) = \begin{cases} e^{-y} & \text{if } y>0 \\ 0 & \text{if } y \le 0 \end{cases}$

Then $f_{XY}(x,y) = h(x)\,g(y)$, for all (x,y).

$\Rightarrow$ X, Y are independent r.v.'s. $\boxtimes$

3) A, B, C.

(a) $\Omega = \{1, 2, \dots, 6\}$

$\mathbb{P}(A) = \frac{1}{2}$, $\mathbb{P}(B) = \frac{1}{6}$, $\mathbb{P}(C) = \frac{2}{6} = \frac{1}{3}$.

$\mathbb{P}(A \cap B) = 0$.

$\mathbb{P}(A \cap C) = \mathbb{P}(5) = \frac{1}{6} = \frac{1}{2} \times \frac{1}{3}$

$A \cap C = \{5\}$ A, C are inependent

$\textcircled{2}$

A, B, C are ind. iff

$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$, $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$

$\mathbb{P}(BC) = \mathbb{P}(B)\mathbb{P}(C)$, and $\mathbb{P}(A \cap B \cap C) =$

$\mathbb{P}(A) \times \mathbb{P}(B) \times \mathbb{P}(C)$

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3) $f_{X,Y}(x,y) = \begin{cases} 2e^{-x-y} & \text{if } x > y > 0 \\ 0 & \text{otherwise.} \end{cases}$

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad f_Y(y) > 0.$$

$-\infty < x < \infty$.

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} \quad \text{if } f_X(x) > 0.$$

$-\infty < y < \infty$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$f_X(x) = 0$ if $x \le 0$, For $x > 0$

$$f_X(x) = \int_0^x f_{XY}(x,y)\,dy = \int_0^x 2e^{-x-y}\,dy$$

$$= 2e^{-x}\left(-e^{-y}\Big|_{y=0}^{x}\right) = 2e^{-x}(1-e^{-x})$$

Hence, for $x>0$

$$f_{Y|X=x}(y) = \begin{cases} \dfrac{2e^{-x-y}}{2e^{-x}(1-e^{-x})}, & \text{if } x>y>0 \\ 0 & \text{if } y>x, \text{ or } y\le 0. \end{cases}$$

$$f_{Y|X=x}(y) = \begin{cases} \dfrac{e^{-y}}{(1-e^{-x})} & \text{if } y\in(0,x) \\ 0 & \text{otherwise}. \end{cases}$$

4) $Cov(aX,Y) = \mathbb{E}\left[(aX-\mathbb{E}(aX))\times(Y-\mathbb{E}Y)\right]$

$= \mathbb{E}\left[(a(X-\mathbb{E}(X)))\times(Y-\mathbb{E}(Y))\right]$

$= a\,\mathbb{E}\left[(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\right] = a\,Cov(X,Y).$

$Cov((X+a),Y) =$

$\mathbb{E}(\underbrace{X+a-\mathbb{E}(X+a)}_{\mathbb{E}(X)+a})\times(Y-\mathbb{E}(Y))$

$= \mathbb{E}(X-\mathbb{E}(X))\times(Y-\mathbb{E}(Y)) = Cov(X,Y).$