SEMINAR TUTORIAL

Dynamical Systems (MTH744U/P)

Semester A 2023-24

QMUL

WEEKS 1-9

ST 1.1

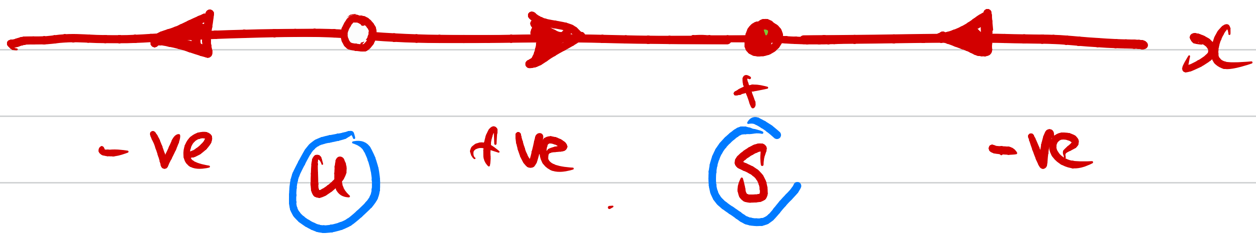
$\dot{x} = 1 - x^{14} = f(x)$

Zeroes of $f(x)$ are fixed pts of $\dot{x} = f(x)$

$1 - x^{14} = 0 = (1 - x^7)(1 + x^7)$

$\Rightarrow x^7 = 1$, $x^7 = -1$ $\Rightarrow$ $x = 1$, $x = -1$

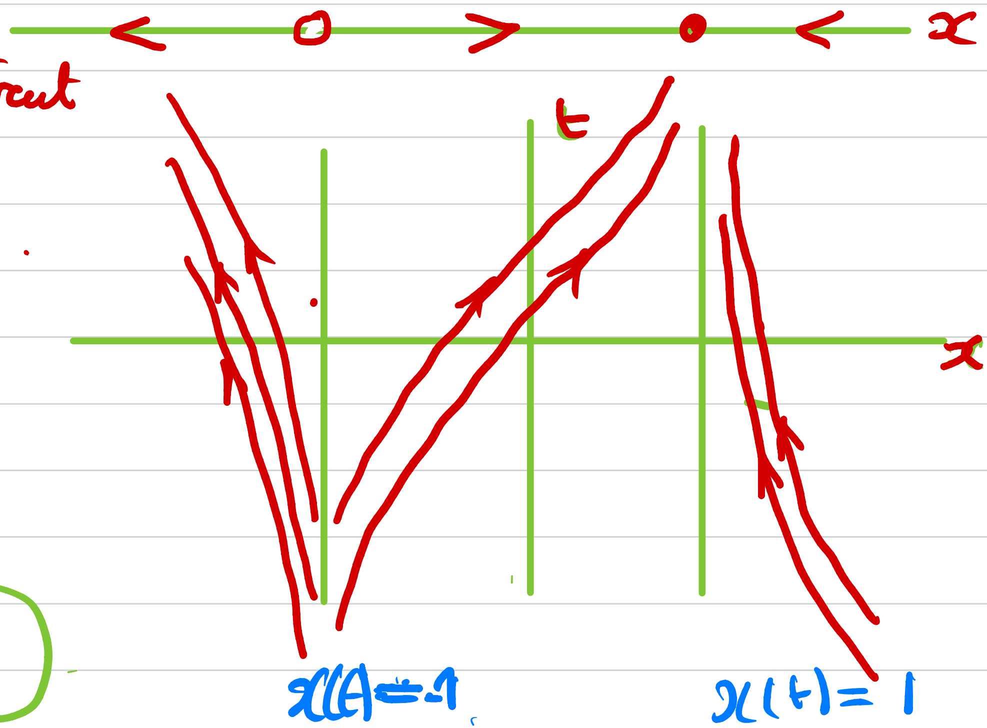
other roots complex



ST1.2

What might the solutions curves look like in the $x-t$ plane? ($\dot{x} = \sin x$)

Phase portrait → $x-t$ diagram.



$\dot{x} = 1 - x^{14}$

$x = x(t)$

$x_0 = -1$

$t_0 = 0$

x $x(t) = -1$

$x(t) = 1$

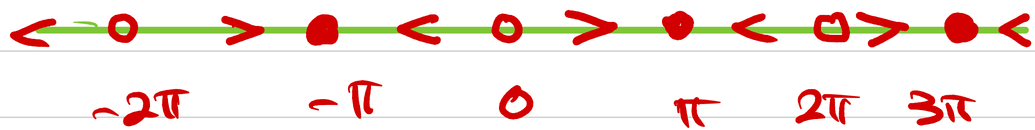
(x_0, t_0)

$\dot{x} = x$?

ST 1.3

$\dot{x} = e^{-x}\sin x$ S1 $\qquad$ $\dot{x} = \sin x$ S2

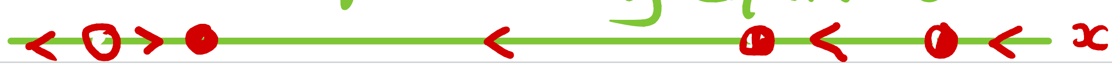
$e^{-x} > 0 \quad \forall x \in \mathbb{R}$ "pluses & minuses of $e^{-x}\sin x$ and $\sin x$ are the same intervals



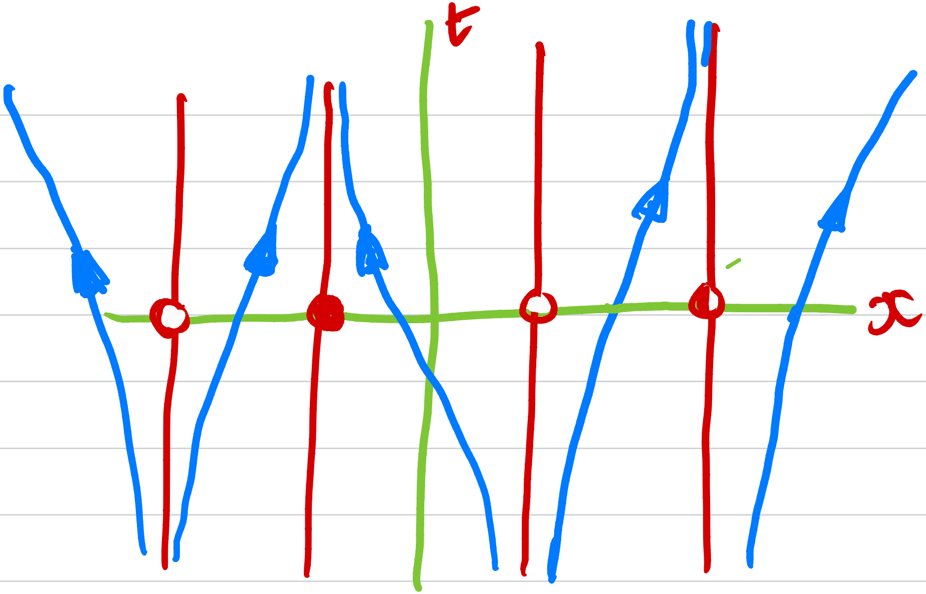
S1 and S2 are qualitatively equivalent
with the same fixed points

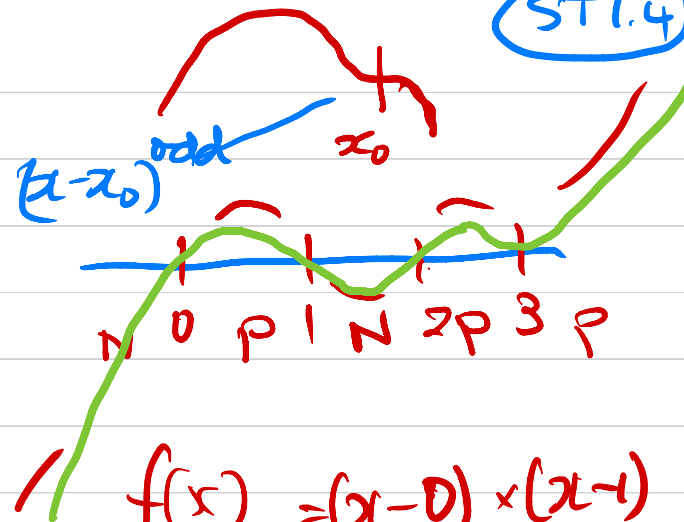
S3


S3 & S4 are qualitatively equiv (fixed points different values)

S4


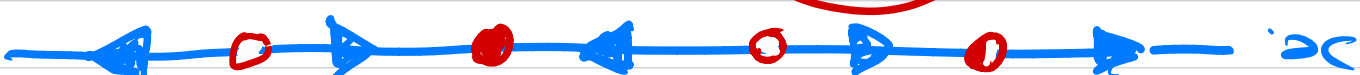
ST1.4





$$f(x) = (x-0)\times(x-1)^2 (x-2)\times(x-3)$$

$\dot{x} = ?? = f(x)$ Q2



Q1 ??

ST1.5

$\dot{x} = f(x)$, f is a ∞-differentiable not necessarily poly or power series

$f(x) = 0, \quad x \le 0$

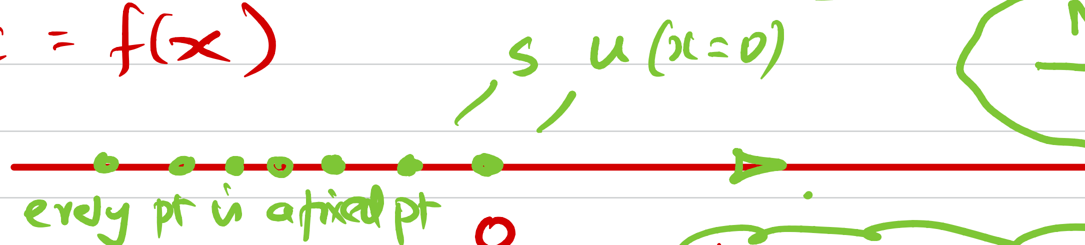
$= e^{-1/x^2}, \quad x > 0$

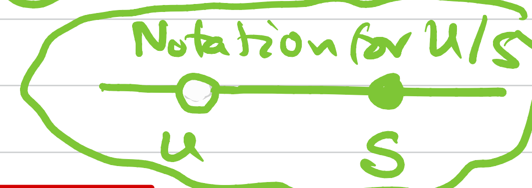
$f'(0) = 0$

$f^{(n)}(0) = 0$

> 0

$\dot{x} = f(x)$



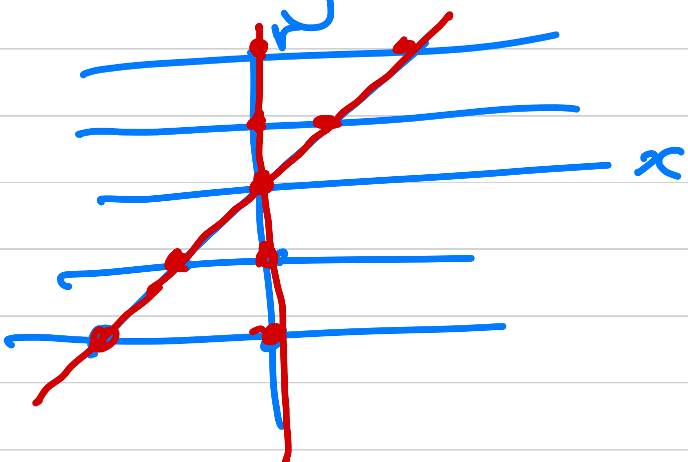


ST2.1

The eqns for this saddle-node coelesence in 2D is given

$$\left.\begin{aligned} \dot{x} &= r + x^2 \\ \dot{y} &= y \end{aligned}\right\} \text{ system in Chapter 4/5}$$

Decoupled eqns, each can be addressed independently



Bifurcatn diagram.

Plot the bifurcation diagram for

$$\dot{x} = xr + x^2$$

ST2.2

BD

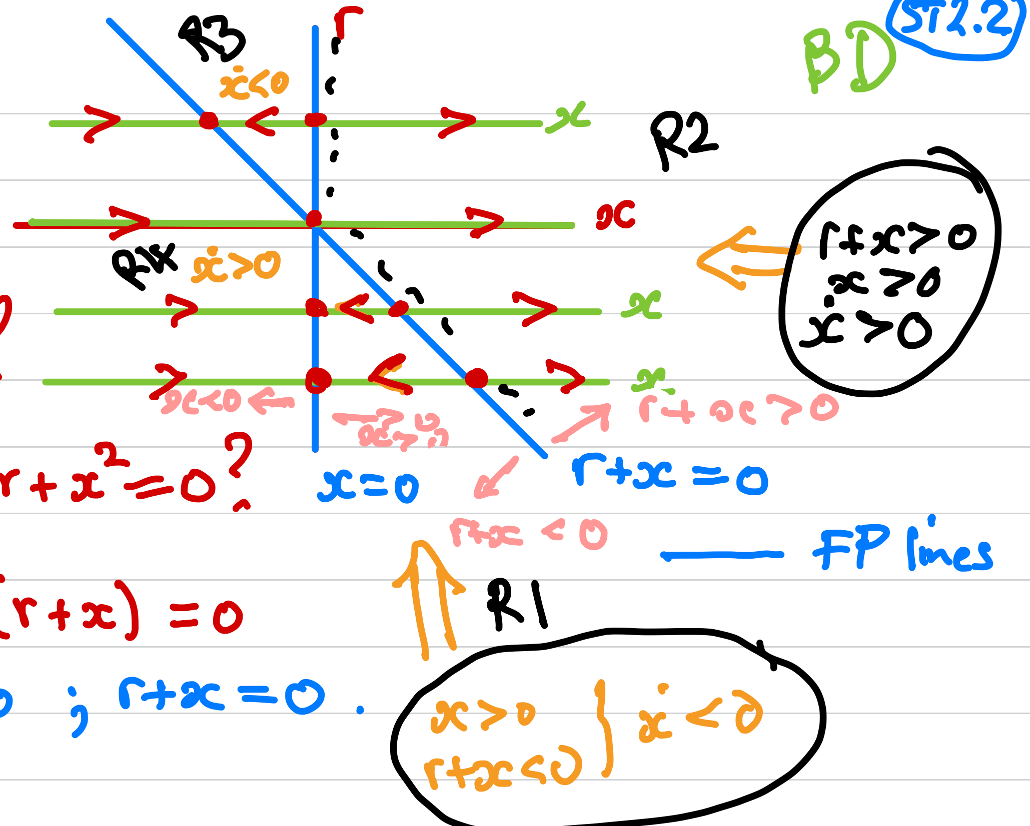
$\dot{x} = xr + x^2$

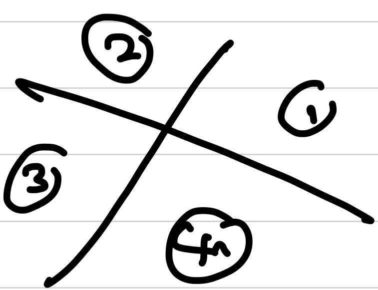
FPs of $\dot{x} = xr + x^2$?

$\dot{x} = 0 \iff xr + x^2 = 0$?

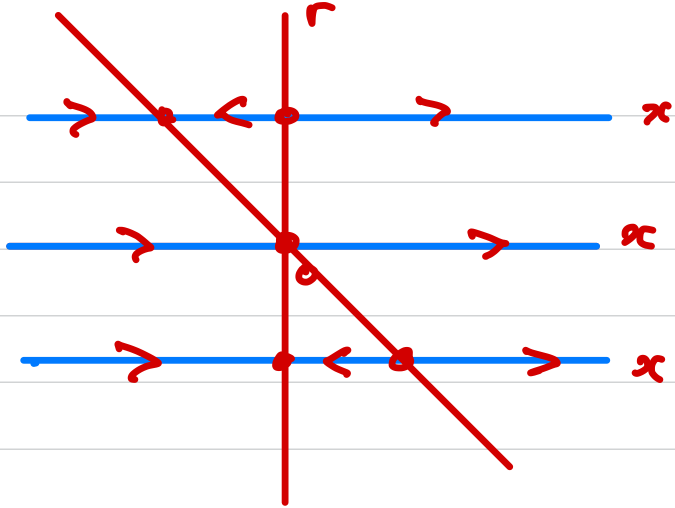
$x(r+x) = 0$

$x = 0$; $r + x = 0$.





ST2.3



This Bifurcation diagram shows the 3 qualitatively distinct phase portraits for the system

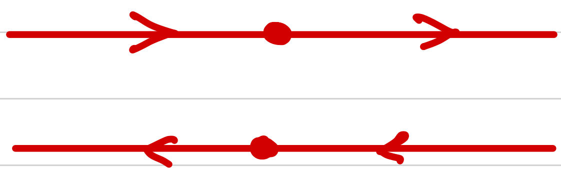
$$\dot{x} = rx + x^2$$

$2 \rightarrow 1 \rightarrow 2$

A : $\dot{x} = x^2$ B: $\dot{x} = -x^2$

A

B



Consider a change of coordinates : let $y = -x$ for system B

$$\dot{y} = -\dot{x} = -(-x^2) = x^2 = y^2 \quad (A)$$

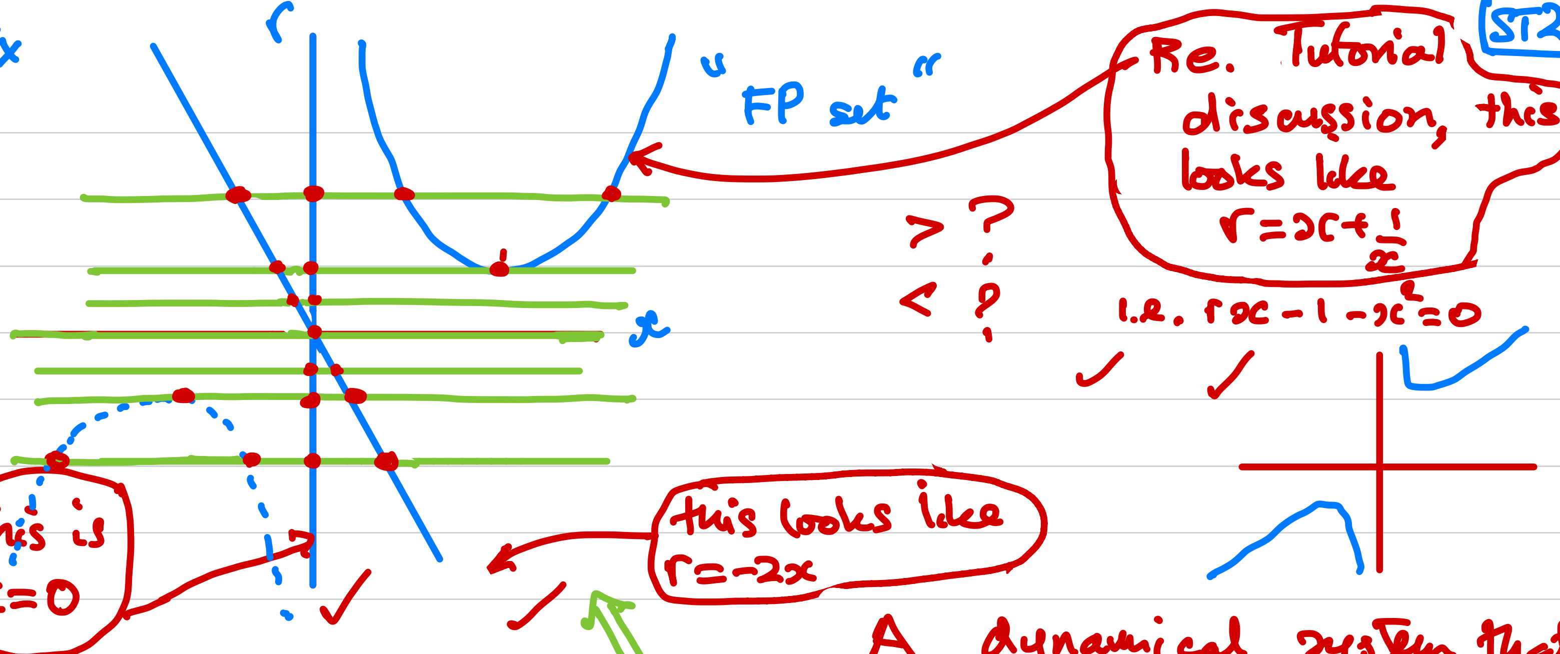
$$\dot{y} = y^2$$

(recall, $\dot{x} = r - \underline{x^2} \implies$

$\dot{z} = r + z^2$.

Ex

ST2.4



What are the distinct qualitative types given the FP set.

We can partially answer without knowing the actual dynamic $\dot{x} = ?$

A dynamical system that fits all the FP criteria is

$$\dot{x} = x(x+r)(rx-1-x^2)$$

Check it out!

provides extra FPs in Q3

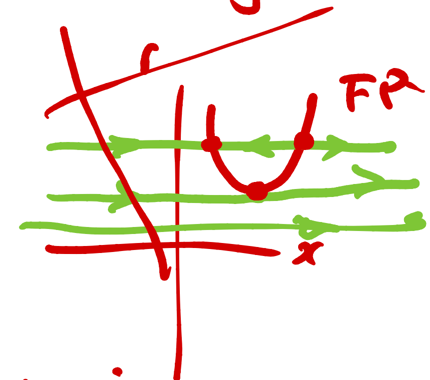
TS 3 Qn 3.1.3 p 80

3.1

$\dot{x} = r + x - \ln(1+x) = f(x)$

Saddle-node bifurcation?

Typical



FP $r + x - \ln(1+x) = 0 \quad *$

$(f(x) = 0)$

BP $1 - \frac{1}{1+x} = 0 \quad **$ $\Rightarrow$ $\left.\begin{array}{l} 1+x = 1 \\ x = 0 \end{array}\right\}$ subs in $*$

$\left(f'(x) = 0 \quad \frac{\partial f}{\partial x} = 0\right)$ $r + 0 - \ln(1+0) = 0$

$r = 0, \ x = 0$ · potential bifurcation point.

3.2

Expand for $x = 0$

$$\dot{x} = r + x - \ln(1+x)$$
$$= r + x - x + \frac{x^2}{2} - \frac{x^3}{3} + \dots$$
$$= r + \frac{x^2}{2} + \text{H.O.T}$$

if $x=2$ fixed pt, $x = 2 + y$ — local coord

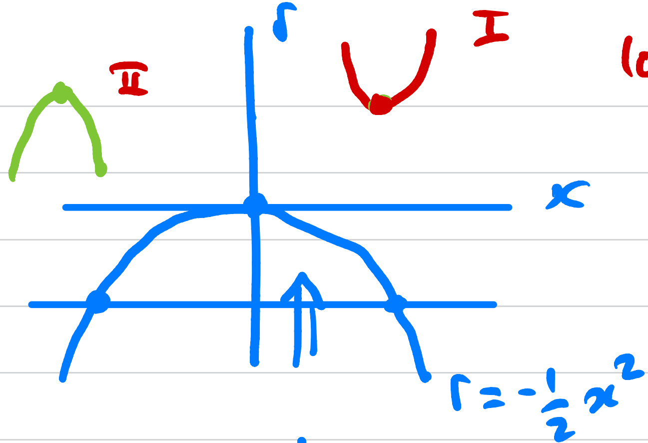
→ change this to $\dot{y} = \mu + y^2$

$$A = \frac{\partial f}{\partial r}\bigg|_{BP} \neq 0 \quad , \quad B = \frac{\partial^2 f}{\partial x^2}\bigg|_{BF} \neq 0$$

$A = 1$, $B = \frac{1}{2}$

∴ Saddle node bifurcation

Qn Subcritical or supercritical Saddle-node bifurcation?



locally

$$\dot{x} = r + \frac{1}{2}x^2$$

FP set $\quad r + \frac{1}{2}x^2 = 0$

$$r = -\frac{1}{2}x^2$$

subcritical ($2 \to 1 \to 0$ with increasing r)
supercritical $0 \to 1 - 2$ ")

3.1.1 $\quad \dot{x} = 1 + rx + x^2$

FPs eqn: $\quad 1 + rx + x^2 = 0$

BF eqn: $\quad r + 2x = 0 \Rightarrow r = -2x$

$\therefore 1 + (-2x)x + x^2 = 0, \quad 1 - x^2 = 0$

$x = \pm 1 \Rightarrow r = \mp 2.$

Bifurcation points:

$\therefore$ $\boxed{x=1, \quad r=-2}$

and $x=-1 \quad r=2$

$(x,r)=(1,-2)$

① We have

$$\dot{x} = 1 + rx + x^2. \quad = f(x,r)$$

② Introduce local coordinates

$$\boxed{\begin{aligned} y &= x - 1 \\ \mu &= r + 2 \end{aligned}} \text{ at } (x,r)=(1,-2)$$

$$\dot{y} = g(y,\mu)$$

$$\dot{y} = \dot{x} = 1 + rx + x^2$$

$$= 1 + (\mu-2)(1+y) + (1+y)^2$$

$$= \cancel{1} + \mu \cancel{-2} + \mu y - 2y + \cancel{1} + \cancel{2y} + y^2$$

$$\boxed{\dot{y} = \mu + \mu y + y^2}$$

$$A = \frac{\partial g}{\partial \mu}\Big|_0 = \frac{\partial f}{\partial r}\Big|_{(1,-2)}$$

$$B = \frac{\partial^2 g}{\partial y^2}\Big|_0 = \frac{\partial^2 f}{\partial x^2}\Big|_{(1,-2)}$$

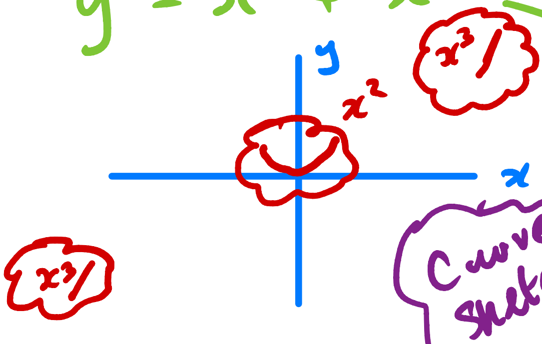
$$C = \frac{\partial^2 g}{\partial y \partial r}\Big|_0 = \frac{\partial^2 f}{\partial x \partial r}\Big|_{(1,-2)}$$

$\underline{A \neq 0}$, $C \neq 0$, $\underline{B \neq 0}$

$\therefore$ Saddle-node

3.5

$y = x^2 + x^3$



Note x^2 dominates close to 0,
x^3 dominates close to '∞'
i.e., $x \to \infty$

Curve sketch note!

3.1.2 $\dot{x} = x(r - e^{x})$, but lets try:

$$\dot{x} = x(r - e^{x} - e^{-x}) = x(r - 2\cosh(x))$$

$x \equiv 0.$
$r = e^{x} + e^{-x}.$
$= 2\cosh(x)$

FPset: $x(r - e^{x} - e^{-x}) = 0$

BP cond: $\frac{\partial}{\partial x}\big(x(r - 2\cosh(x))\big) = 0$

$\Downarrow$

BPcond: $x(-2\sinh(x)) + (r - 2\cosh(x))$

3.6

$$-2x\sinh(x) + r - 2\cosh(x) = 0 \left.\vphantom{\begin{matrix}a\\b\\c\end{matrix}}\right\} \begin{matrix}\text{BP}\\ \text{FP.}\end{matrix}$$

$$r = 2\cosh(x)$$

$$-2x\sinh(x) = 0$$

$$\Rightarrow \quad x = 0$$

$x = 0, \quad r = 2$ (BP?)

Expand abt

$x, \quad \mu = r - 2 \Rightarrow r = \mu + 2$

$$\dot{x} = x(r - 2\cosh(x))$$

$$= x\left(r - 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\right)$$

$$= x\left(r - 2 - \frac{x^2}{2!}\right) = x\mu - \frac{x^3}{2!}$$

Not a saddle node

$A = B = 0, C \neq 0$

WK4

4.1

$$\dot{x} = x^3 + (1-\mu)x^2 - (1+\mu)x + \mu^2 - 1 = f(x,\mu)$$

FPs

$$x^3 + (1-\mu)x^2 - (1+\mu)x + \mu^2 - 1 = 0$$

$$(x - \mu + 1)(x^2 - (1+\mu)) = 0$$

FPs

$$(x - \mu + 1)(x^2 - (1+\mu)) = 0$$

TWO fixed points curves.

FP $x = \mu - 1$

Linear stability

$$\frac{\partial f}{\partial x}(x,\mu)\Big|_{x=\mu-1} \begin{array}{l} >0 \ \ u \\ <0 \ \ s \end{array}$$

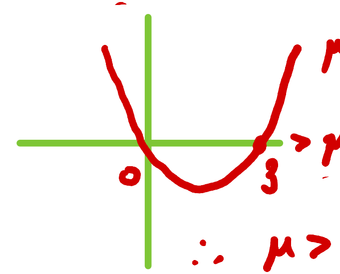
Let's check

$$\frac{\partial f}{\partial x} = 3x^2 + 2x(1-\mu) - (1+\mu) \Big|_{x=\mu-1}$$

$= g(\mu)$, quadratic in μ

$= 3(\mu-1)^2 - 2(\mu-1)^2 - (1+\mu)$

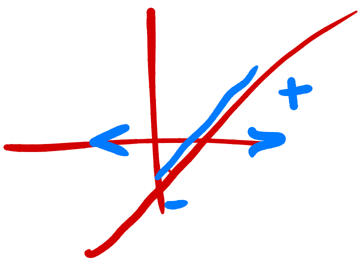
$= \mu^2 - 3\mu$



At $x=\mu-1$:

$\therefore$ $\left.\begin{array}{l}\mu>3\\ \mu<0\end{array}\right\}$ unstable

$0<\mu<3$ stable



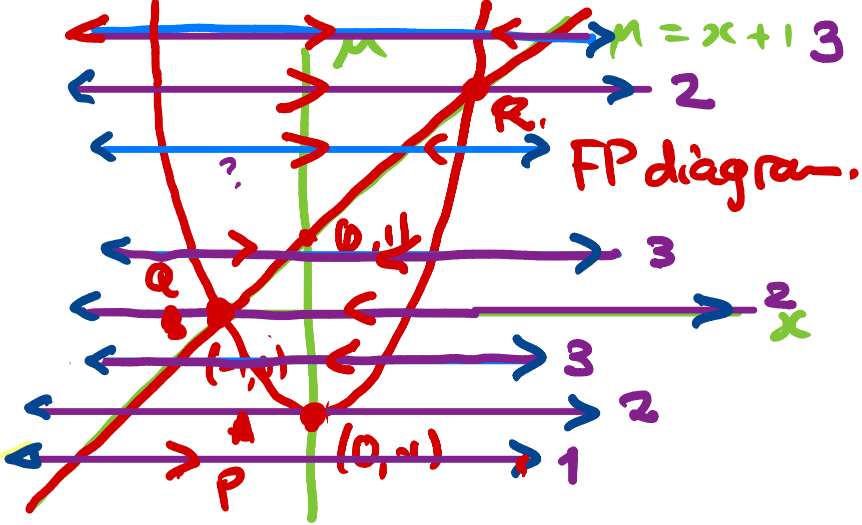
$$\boxed{x = \mu - 1} \quad C_1$$

$$\boxed{x^2 = 1 + \mu} \quad C_2$$

curve of fixed pts

note change - a mistake!

Linearly UNSTABLE at (x,μ) if	$\left.\dfrac{\partial f}{\partial x}\right\|_{(x,\mu)}$	>0
" STABLE at (x,μ)		<0



$P = (0,-1)$

$Q = (-1,0)$

$R = (2,3)$

Bifurcation points

$x = \mu - 1$

$\mu = x + 1$

$x^2 = \mu + 1$

$\mu = x^2 - 1$

$x^2 = \mu + 1$ and $x = \mu - 1$

$\therefore (\mu - 1)^2 = \mu + 1$

$\mu^2 - 2\mu + 1 = \mu + 1$

$\Rightarrow \mu^2 - 3\mu = 0 \Rightarrow \mu(\mu - 3) = 0$

$\therefore \mu = 0, \; x = -1$ (Q)

$\mu = 3, \; x = 2$ (R)

Exam Taylor expansion at points P, Q, R. 4.4

to obtain the form of the bifurcation

form form

P , $A, B \neq 0$ SNB ?

Q , $B, C \neq 0$, $A = 0$ TB ?

R " " TB. ?

Tutorial Week 5

T5.1

Q1. $\dot{x} = \sin^2 x - \tan^2 x = f(x)$

$$f(x) = \frac{\sin^2 x}{\cos^2 x}(\cos^2 x - 1) = \frac{-\sin^4 x}{\cos^2 x}$$

$\sin x \approx x$ for small x $\therefore$ $\sin^4 x \approx x^4$ and $\cos x \approx 1$

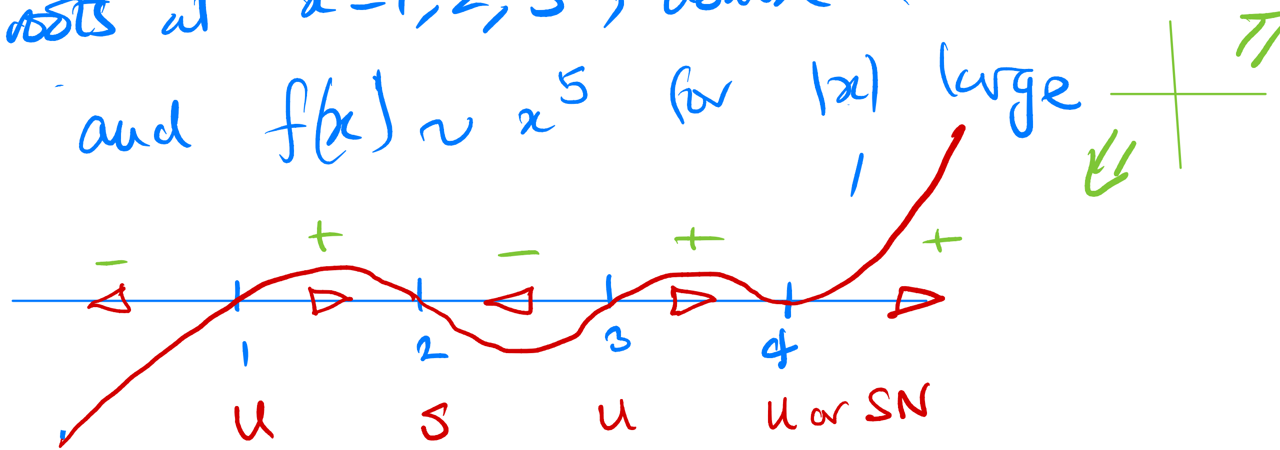
$\therefore$ At the origin $\dot{x} = -x^4$, which is of saddle node type or unstable. (both correct)

T5.2

Q2 f(x) has four roots and we need to investigate each of the corresponding dynamical systems for stability, instability

example $f(x) = (x-1)(x-2)(x-3)(x-4)^2$.

Single roots at $x = 1, 2, 3$, double roots at $x=4$ and $f(x) \sim x^5$ for $|x|$ large



Q3 $\dot{x} = -rx + x(1+2x^2)/(1+3x^2) = f(x,r)$ T5.3

Note $x=0$ is a fixed point $\forall r$.

We could find change of stability by calculating $f'(x,r)$ but let's not!

Expand in local coord x (at $x=0$!)

$$\dot{x} = -rx + x(1+2x^2)(1+3x^2)^{-1}$$

$$= -rx + x(1+2x^2)(1-3x^2) \dots$$

$$= -rx + x(1-x^2+O(x^3)) \dots$$

$$= (r+1)x - x^3 + O(x^3)$$

$$= \mu x - x^3$$

with local coordinates

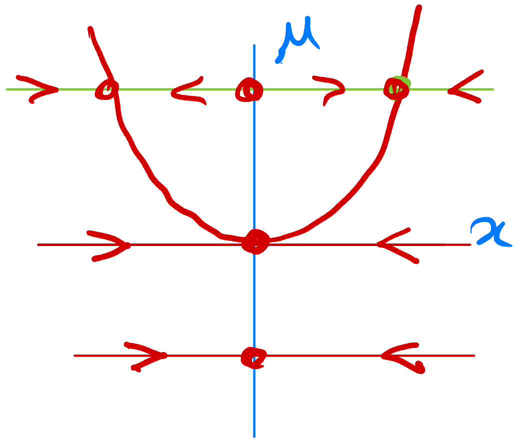
$\mu = 0$ $(r=1)$

$x = 0$

T5.4

So in (x, μ) coordinate $\quad x = \pm\sqrt{\mu}$, $\mu \geq 0$

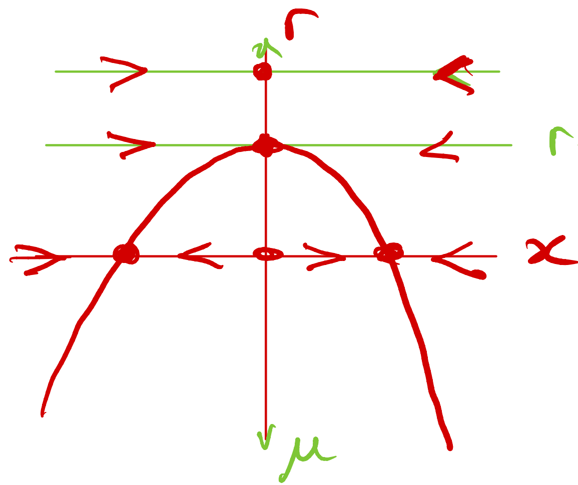
& $x = 0$, $\forall \mu$.



Supercritical pitchfork. in (x,μ) coords

<u>but</u> $\mu = 1 - r$

i.e $r = 1 - \mu$



Subcritical in (x, r) coordinates at $(0,1$

T5.5

Q4 $\dot{x} = rx + \dfrac{x(1+x)}{(1+x+3x^2)}$

Again $x=0$ is a FP $\forall$ $x=0$

Expand in power series about $x=0$.

$$\dot{x} = rx + x(1+x)(1+x+3x^2)^{-1}$$

$$= rx + x(1+x)(1-(x+3x^2)+(x+3x^2)^2-\ldots$$

$$= rx + x(1+x)(1-x-3x^2+x^2+O(x^3))$$

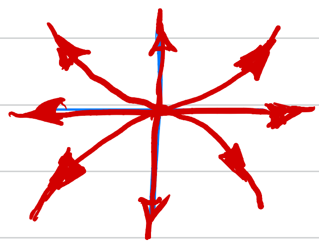
$$= rx + x(1+x)(1-x-5x^2)$$

$$= rx + x(1-x^2-3x^2+x^2)+\ldots = (r+1)x - \underline{3x^3}$$

$\therefore$ Supercritical pitchfork at $x=0$, $r=-1$

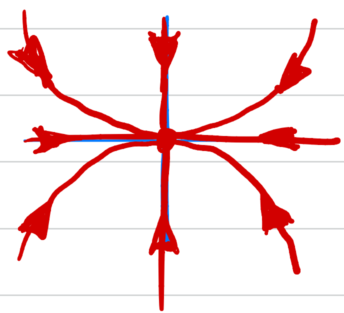
Q5. I did just a few quick examples as we began discussing chap 4 in the last 10 minutes of Tuesday's Lecture.

$\dot{x} = x$
$\dot{y} = 2y$



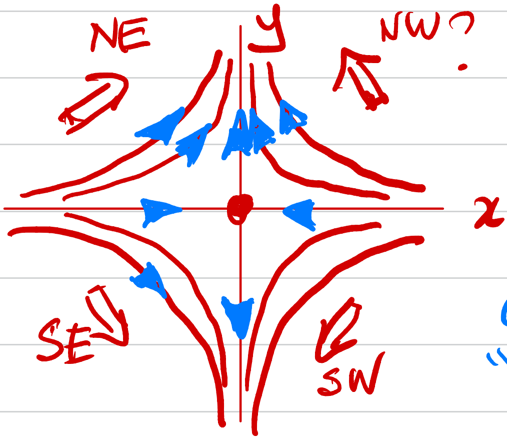
"unstable node"

$\dot{x} = -x$
$\dot{y} = -2y$



"stable node"
also Asymptotically stable

$$\dot{x} = -x \qquad \frac{-dx}{x} = \frac{dy}{y} \Rightarrow y\,dx + x\,dy = 0$$
$$\dot{y} = +y \qquad \Rightarrow xy = \text{constant (hyperbolas)}$$



Saddle

unstable

orbits escape to the "north" and "south" after a closest approach to (0,0)
(Astronomer in me!)

Matter of interest!

the Basin of Attraction of the fixed pt at (0,0) is the $\underline{x\text{-axis}}$!!

Tutorial wk 6

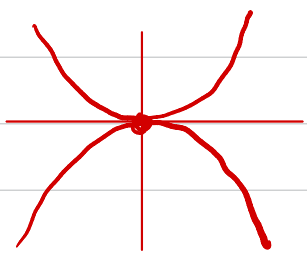
$$\dot{x} = \lambda_1 x \qquad A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$\dot{y} = \lambda_2 y$$

$$\frac{dx}{\lambda_1 x} = \frac{dy}{\lambda_2 y}$$

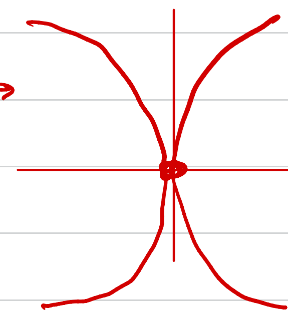
$$\Rightarrow \quad y = C x^{\mu}, \quad \mu = \frac{\lambda_2}{\lambda_1}$$

$\frac{\lambda_2}{\lambda_1} > 1 \rightarrow$

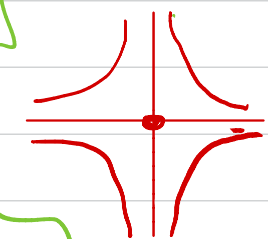


$0 < \frac{\lambda_2}{\lambda_1} < 1 \rightarrow$

axes orbits



$\lambda_2 / \lambda_1 < 0$

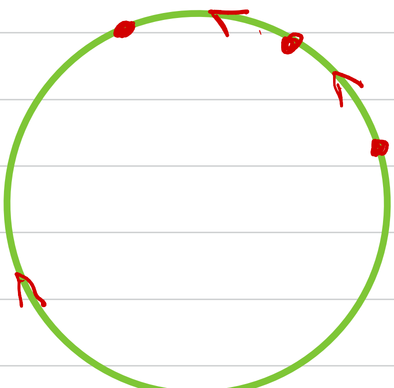


time orientation $\begin{cases} \lambda_2, \lambda_1 < 0 & \text{- stable} \\ \lambda_2, \lambda_1 > 0 & \text{- unstable} \end{cases}$

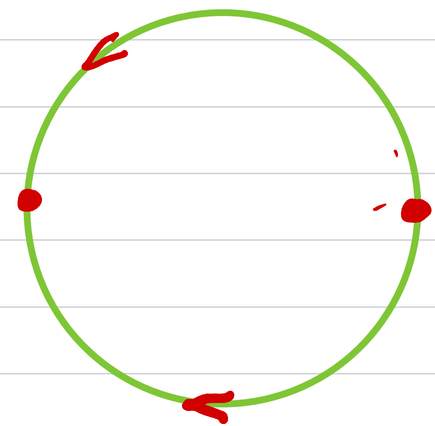
T6.2

Constructing systems on the circle with given dynamical properties!

$$\dot{\theta} = \sin\theta$$



$\neq$



Modify.

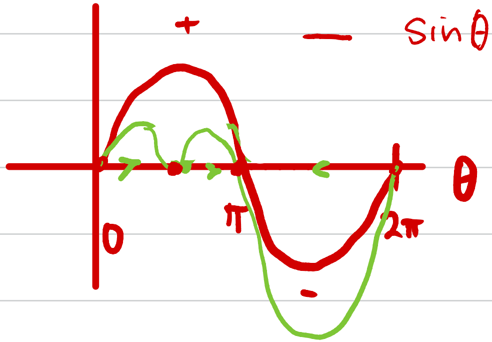
??

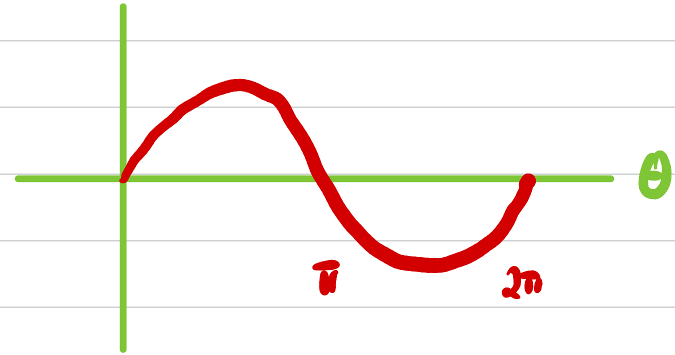
T6.3

— $\sin\theta(1-\sin\theta)$. $\overset{0}{\bullet}$ + $\bullet$ + $\bullet$ — $\overset{2\pi}{\bullet}$

— $\sin\theta$ $\overset{0}{\bullet}$ + $\bullet$ — $\overset{2\pi}{\bullet}$

$\dot\theta = \sin(\theta)$?





Modify to $\dot\theta = \sin(\theta)(1-\sin\theta)$

Given $1-\sin\theta$ is ≥ 0

and only $=0$

for $\theta = \pi/2$, we get $\text{sgn}(\sin\theta(1-\sin\theta)) = \text{sgn}(\sin\theta)$

except for $\theta = \pi/2$ when $\dot\theta = 0$.

So $\times(1-\sin\theta)$ adds a fixed pt at $x = \pi/2$.

T6.4

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}4 & 1\\ 6 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

$Tr = 5$, $Det = -2$

$\lambda^2 + 5\lambda - 2 = 0$, $\lambda = (5 \pm \sqrt{25+8})/2$

$\lambda_1 > 0$, $\lambda_2 < 0$ Saddle fixed point.

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

$Tr = 5$ $Det = 6$

$\lambda_1 = 2$, $\lambda_2 = 3$ unstable node f.p.

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & 3\\ -2 & 1\end{bmatrix}$$

$Tr = 2$, $Det = 7$ $\lambda^2 + 2\lambda + 7 = 0$

$\lambda = \dfrac{2 \pm \sqrt{4-28}}{2} = 1 \pm \sqrt{-6}$

$= 1 \pm \sqrt{6}\, i$ unstable spiral

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

$Tr = 1$, $Det = 0$ $\lambda^2 - \lambda = 0$

$\lambda_1 = 1$, $\lambda_2 = 0$. (see phase portrait next page).

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

$Tr = 0$, $Det = 1$, $\lambda_{1,2} = \pm i$ centre

$\dot{r} = 0$, $\dot{\theta} = 1$ uniform motion on concentric circles.

T6.5

$$\begin{bmatrix}\dot x\\ \dot y\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \qquad \begin{aligned}\dot x &= x+y\\ \dot y &= 0\end{aligned} \qquad \lambda_1 = 1,\ \lambda_2 = 0$$

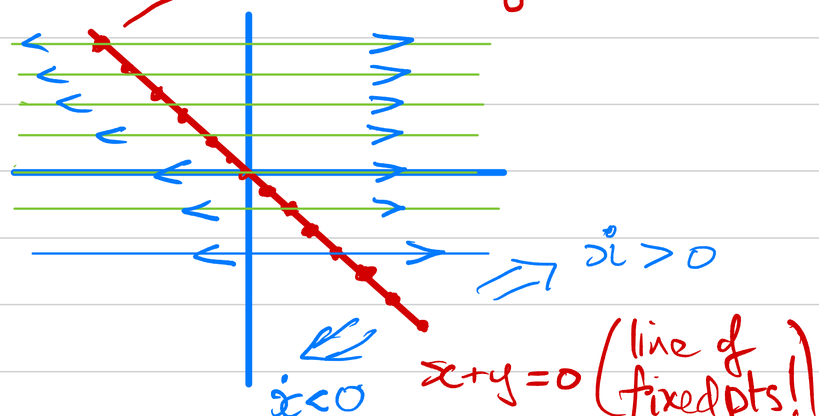
$\lambda_1, \lambda_2 \neq 0$

So the eigenvalues don't fit any of our categories node, spiral, saddle, centres ( )

Investigate. $\dot x = x+y$, $\dot y = 0$

$\dot y = 0 \Rightarrow y(t) = \text{const}$ (horizontal lines in xy-plane. line of fixed points

Also $\dot x = 0$ for $x+y=0$



The origin is an unstable fixed pt.

T8.1

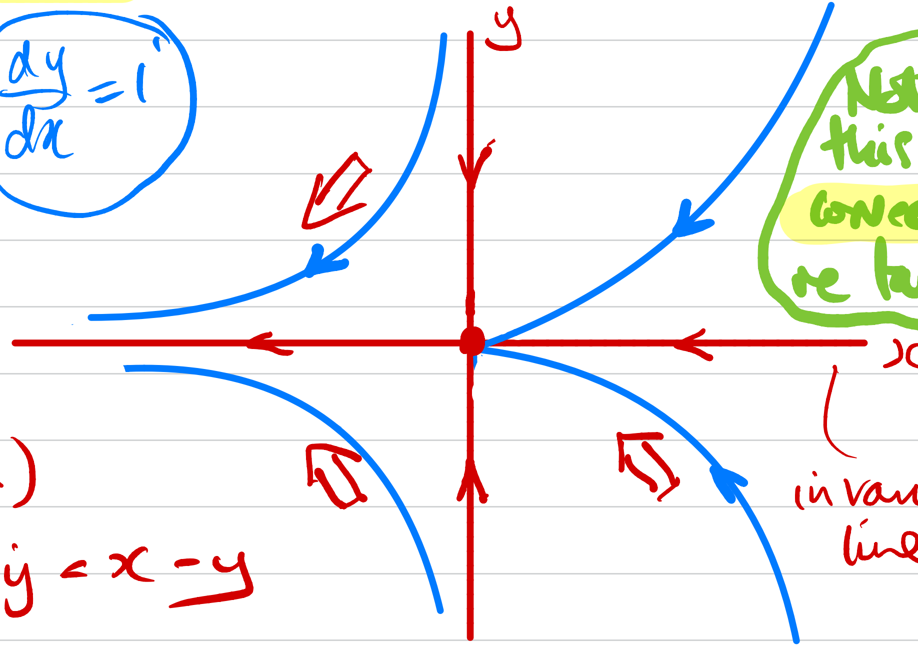
WEEK 8 Tutorial see MIDTERM QUIZ Solutions. on QMPlus

WEEK 9 Tutorial

$\dot{x} = -x^2$

$\dot{y} = -y$

$\frac{dy}{dx} = 1$



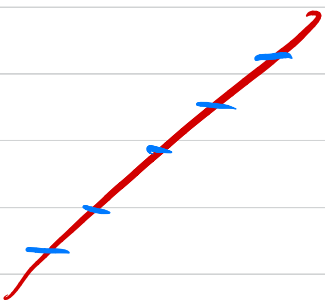
Note this correct re tangent!

invariant line

$\dot{y} = 0$ null-cline

$y = 0$ (no vert motion)

$\dot{y} = x - y$



$$\frac{dx}{-x^2} = \frac{dy}{-y}$$

$$\int \frac{dx}{x^2} = \int \frac{dy}{y} \Rightarrow$$

$$-\frac{1}{x} = \ln(y) + C$$

$$y = C' e^{-1/x}$$

9.2

$y = Ce^{-1/x}$ $\qquad x \to 0^+ \qquad -\frac{1}{x} \to \infty^- \qquad$ & $y \to 0$

Question: What is the tangency of the curve at the origin

Note the slope of the curve is

$$\frac{dy}{dx} = Ce^{-1/x}\left(-\frac{1}{x^2}\right) = \frac{Ce^{-1/x}}{x^2} = \frac{\frac{1}{x^2}}{Ce^{1/x}}$$

$\to \frac{0}{0}$ $\qquad \to \frac{\infty}{\infty}$

Using L'Hopital's Rule (3 times, a personal best for me!)

$$\lim_{x\to 0} \frac{\frac{1}{x^2}}{Ce^{1/x}} = \lim_{x\to 0} \frac{\frac{-2}{x^3}}{Ce^{1/x}\left(-\frac{1}{x^2}\right)} = \lim_{x\to 0} \frac{\frac{2}{x}}{Ce^{1/x}} = \lim_{x\to 0} \frac{\frac{-2}{x^2}}{Ce^{1/x}\left(\frac{-1}{x^2}\right)} = \lim_{x\to 0} \frac{2}{Ce^{1/x}} \to 0$$

$\to \frac{\infty}{\infty}$ $\qquad \to \frac{\infty}{\infty}$

Answer - tangent to the x-axis. See next page there is a late "swerve in the curve" - a pt of inflexion?

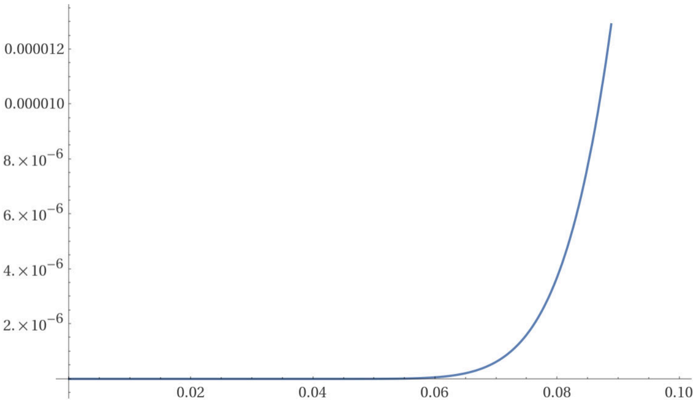
9,3

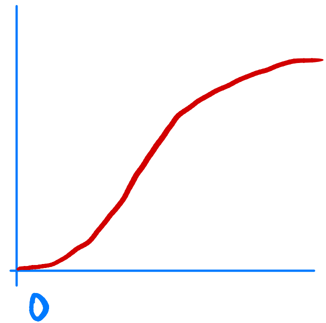
plot y=exp(-1/x); 0<x<0.1

Input interpretation

plot	$y = \exp\left(-\frac{1}{x}\right)$	$x = 0$ to 0.1

Plot





Ex 6.3.1

$$\dot{x} = x - y \qquad \dot{y} = x^2 - 4$$

<u>FPs.</u> $\dot{x} = 0, \quad \dot{y} = 0, \quad \underline{\underline{x = y}}, \quad \underline{x^2 - 4} = 0$

$$x = \pm 2, \quad y = \pm 2$$

$$(x, y) = (2, 2), \quad (x, y) = (-2, -2)$$

$(x, y) = (2, 2)$ Linearisation.

$$J(\underline{x})\Big|_{(2,2)} = \begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$$

$$\dot{\underline{x}} \overset{?}{=} \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix} \underline{x}$$

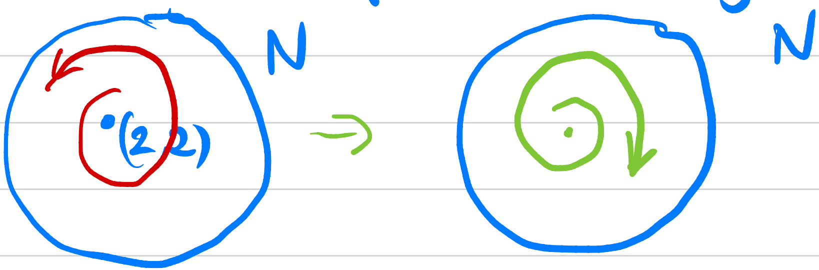
$$(\lambda - 1)(\lambda) + 4 = 0$$

$$\lambda^2 - \lambda + 4 = 0$$

$$x = \frac{1 \pm \sqrt{1 - 16}}{2} = \frac{1 \pm i\sqrt{15}}{2}$$

This is hyperbolic. (non-zero real parts for λ_1, λ_2)

– unstable spiral linearly



9.5

$(x,y) = (-2,-2)$

$$\begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix}.$$

$$(\lambda-1)\lambda - 4 = 0$$

$$\lambda^2 - \lambda - 4 = 0 \qquad \lambda = \frac{+1 \pm \sqrt{1+16}}{2}.$$

$$\lambda_1 = (1+\sqrt{17})/2 \quad > 0$$

$$\lambda_2 = (1-\sqrt{17})/2 \quad < 0$$

Saddle

Eigenvalues $\begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \left(\frac{1+\sqrt{17}}{2}\right)\begin{bmatrix} u \\ v \end{bmatrix}$

$u - v = \left(\frac{1+\sqrt{17}}{2}\right)u$? or $-4u = \left(\frac{1+\sqrt{17}}{2}\right)v$

9.7

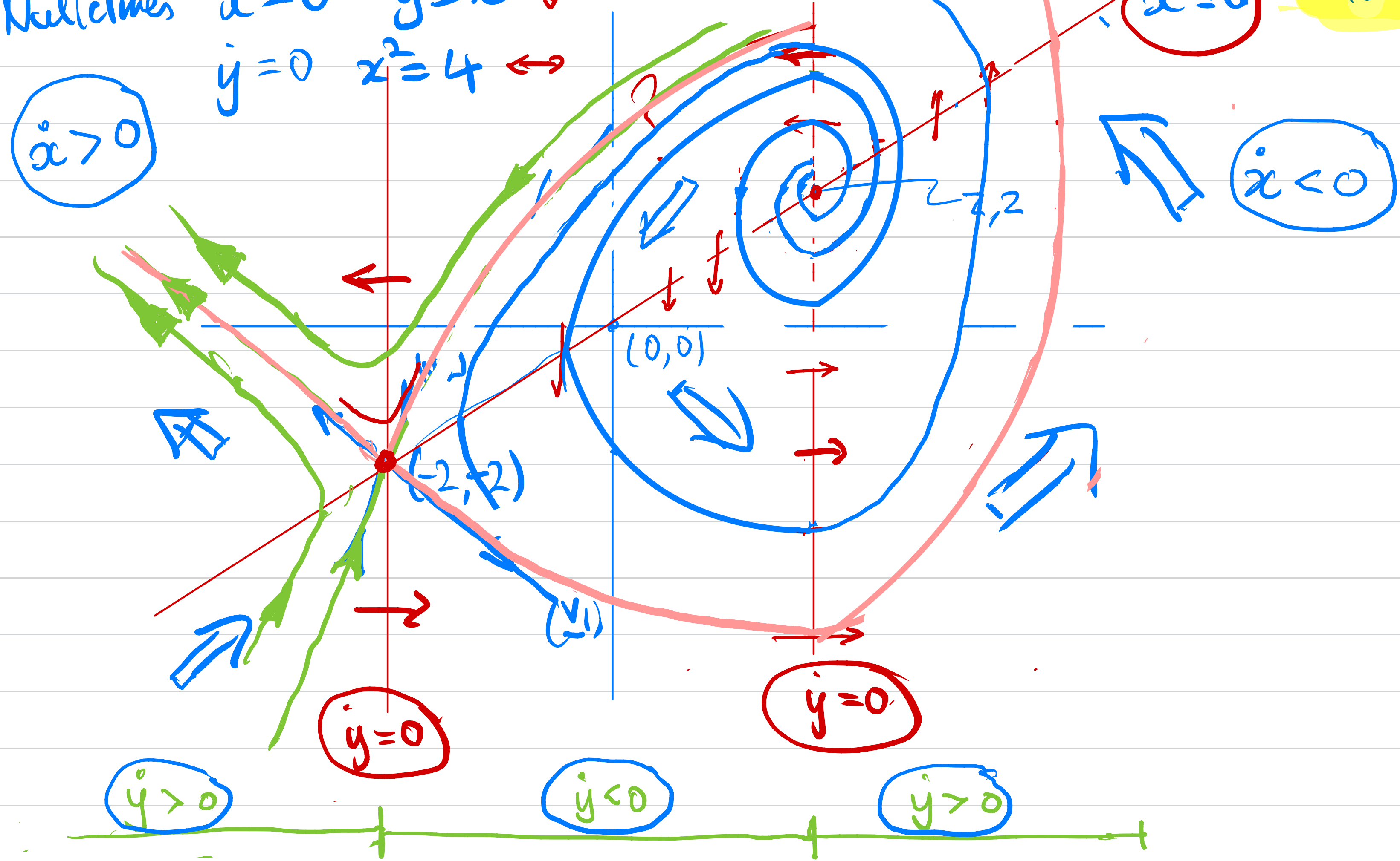
λ_1 : $v = \dfrac{-8u}{(1+\sqrt{17})} \approx \dfrac{-8u}{5+} \approx -1.6u$. $\underline{v_1}$

λ_2 : $v = \dfrac{-8u}{(1-\sqrt{17})} \approx \dfrac{-8}{-3} = 2.7u$ v_2

9.8

Nullclines $\dot{x}=0$ $y=x$

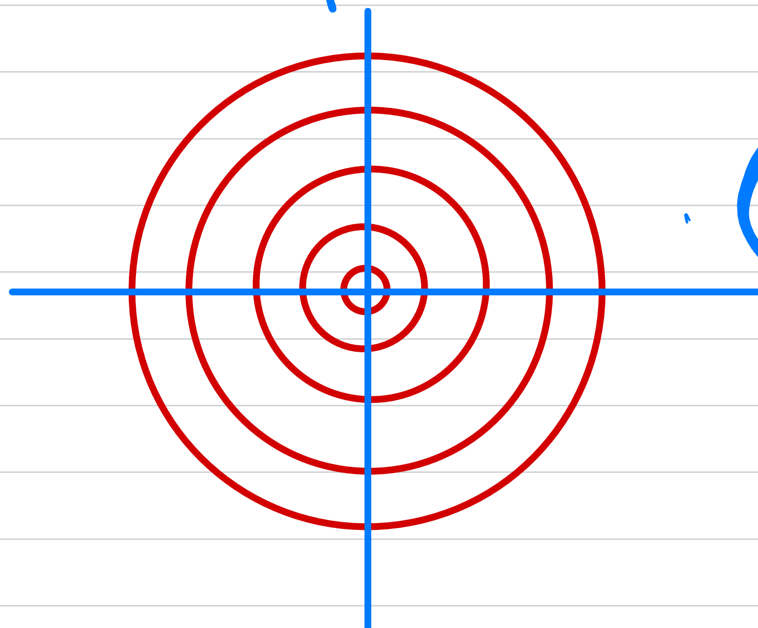
$\dot{y}=0$ $x^2=4$



WEEK 10

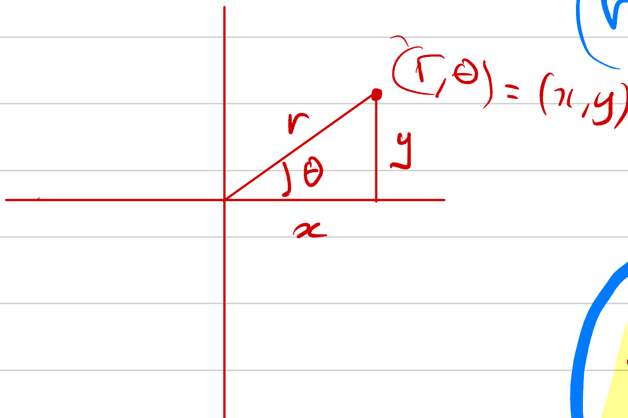
Nonlinear systems in polar coordinates

$\dot{r} = \alpha r \qquad \dot{\theta} = \beta$ ✓ LS



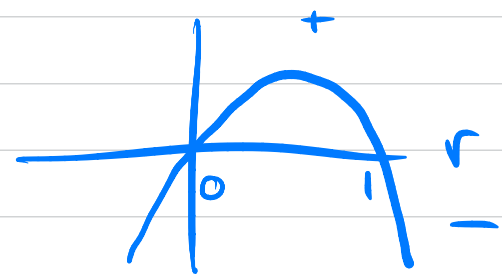
Centre

$$\dot{r} = r(1-r), \quad \dot{\theta} = 1$$

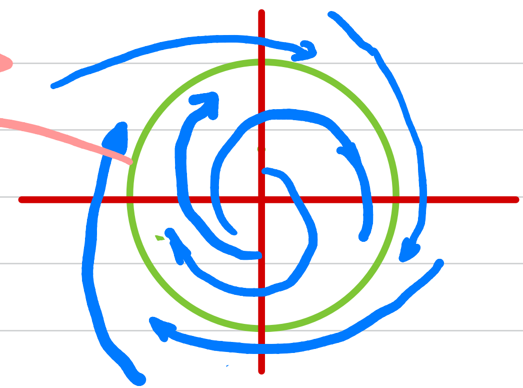


(r, θ)

$\dot{r} = 0$ $\quad r = 0, \; r = 1$



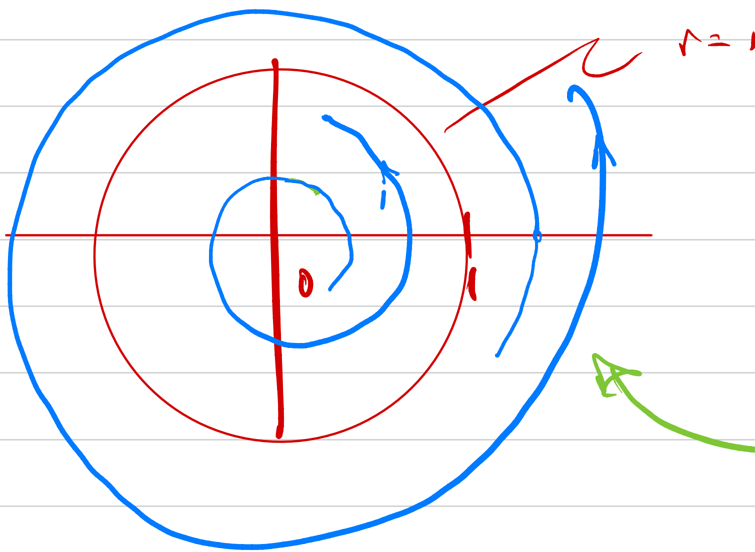
Stable limit cycle

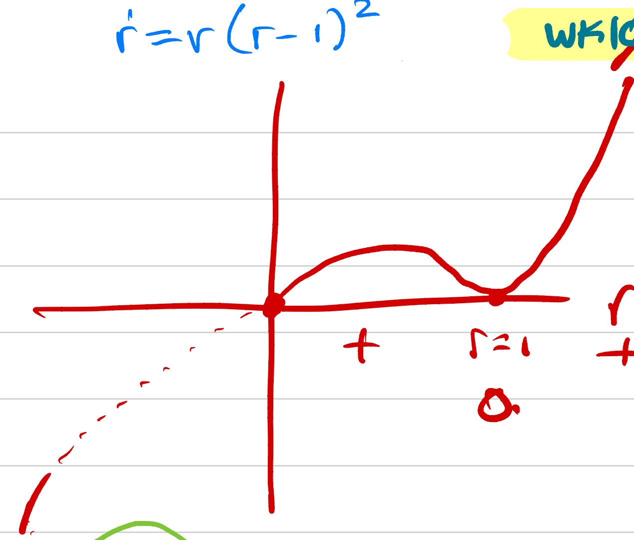


WK10.3

Ex

$\dot{r} = r(r-1)^2, \ \dot{\theta} = 1$





Semi-stable limit cycle.

$$\dot{x} = y - x(x^2+y^2), \quad \dot{y} = -x - y(x^2+y^2)$$

Example

Consider The system

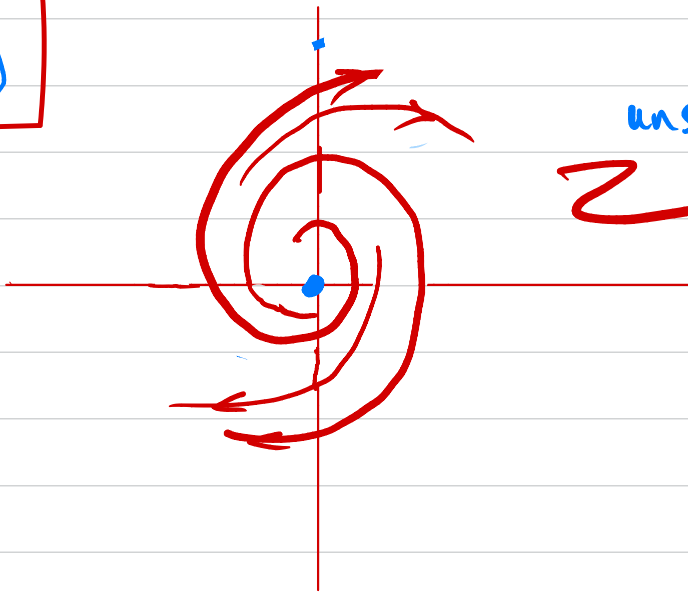
$$\boxed{\begin{aligned}\dot{x} &= y - x(x^2+y^2)\\ \dot{y} &= -x - y(x^2+y^2)\end{aligned}}$$

$$r\dot{r} = x\dot{x} + y\dot{y} \qquad r\dot{r} = xy - x^2r^2 - xy - y^2r^2 = -r^4$$

$$\therefore \; \dot{r} = -r^3$$

$$r^2\dot{\theta} = x\dot{y} - y\dot{x} = -x^2 - xyr^2 - y^2 + yxr^2 = -r^2$$

$$\therefore \; \dot{\theta} = -1$$

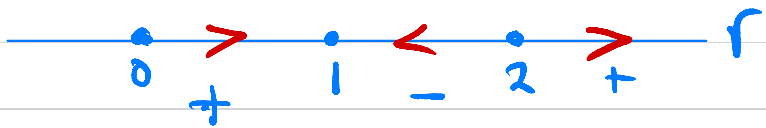


unstable anti-clockwise spiral.

Example

$$\dot{r} = r(r-1)(r-2)$$
$$\dot{\theta} = 3-2r$$

$\dot{\theta} = 3-2r$

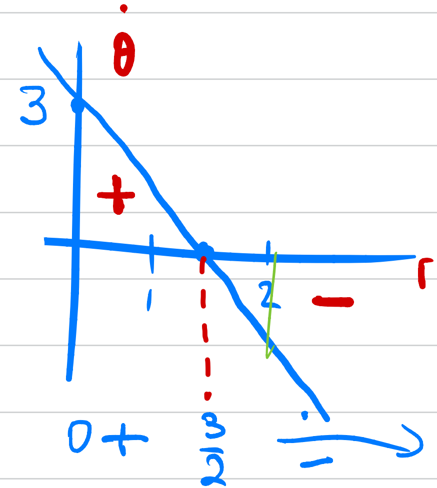


$\dot{\theta} = 3-2r \Rightarrow \dot{\theta} > 0 \quad 0<r<3/2$

$\dot{\theta} < 0 \quad 3/2 < r$

$\dot{\theta} = 0 \Rightarrow$ instantaneous radial motion

$\dot{r} = 0 \Rightarrow$ instantaneous transverse motion



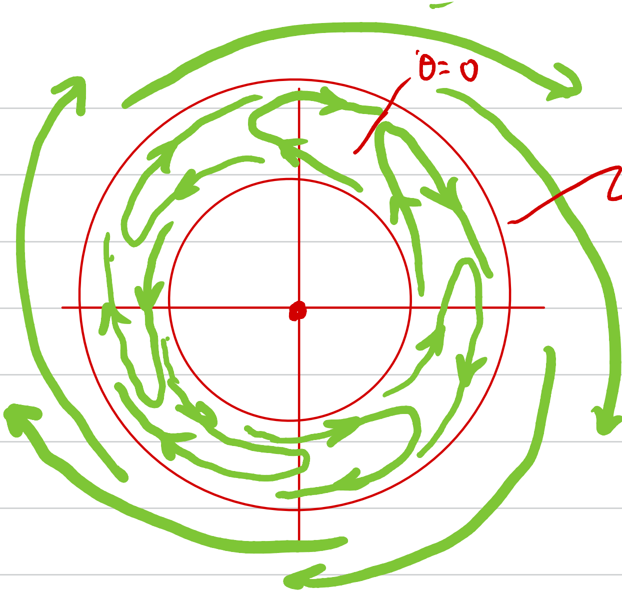
NOTE THE 4 DIRECTIONS IN x, y coords

N S E W

are replaced by

r incr, r decr, clockwise, anti-clockwise

clockwise − anticlockwise for θ



$0<r<1$ $\begin{cases}\dot{r}>0\\ \dot{\theta}>0\end{cases}$

$1<r<3/2$ $\begin{cases}\dot{r}<0\\ \dot{\theta}>0\end{cases}$

$\frac{3}{2}<r<2$ $\begin{cases}\dot{r}<0\\ \dot{\theta}<0\end{cases}$

$2<r$ $\begin{cases}\dot{r}>0\\ \dot{\theta}<0\end{cases}$

Null-clines in polar coordinates are

$\dot{r}=0$, transverse motion

$\dot{\theta}=0$, radial motion

21:52 Thu 30 Nov

Wk10.7

2020

Question 2 [35 marks]. Two dimensional systems

(a) For each of the following systems, find the fixed points and classify them, sketch the null-clines and the vector field, and suggest a plausible phase portrait.

(i)

$$\dot{x} = x + y, \quad \dot{y} = 1 - e^{-x}.$$

[6]

(ii)

$$\dot{x} = x^2 - y, \quad \dot{y} = x - y.$$

[6]

(b) Consider the system

$$\dot{x} = xy, \quad \dot{y} = -x^2.$$

(i) Show that the quantity $E(x,y) = x^2 + y^2$ is conserved over time. [4]

(ii) Show that the origin is not an isolated fixed point. [4]

(iii) Sketch the phase portrait. [3]

(c) A certain two dimensional system is known to have three fixed points, one saddle and two unstable nodes. Sketch a plausible phase portrait which has, as its only periodic orbits, the three fixed points described and a single stable limit cycle. [6]

(d) Find a dynamical system in polar coordinates in the form $\dot{r} = f(r,\theta)$, $\dot{\theta} = g(r,\theta)$, where f, g are suitably chosen functions, which exhibits a planar phase portrait with an unstable spiral focus at the origin surrounded by two circular limit cycles given by $r = 1$ (stable), and $r = 2$ (unstable), with anti-clockwise flow on the inner limit cycle, and clockwise flow on the outer limit cycle. [6]

$\dot{x} = 0, \; \dot{y} = 0$

$3x\dot{x} + y\dot{y}$

T.(1).1

(b) Consider the system of differential equations

$$\dot{x} = x(1 - 3x^2 - y^2) - y(1 + x), \;\; \dot{y} = y(1 - 3x^2 - y^2) + 3x(1 + x). \tag{4}$$

(i) Compute the fixed points of the system (4). For each fixed point determine the stability using linear stability analysis. [8]

(ii) Consider the quantity $L = (1 - 3x^2 - y^2)^2$. Show that $\frac{dL}{dt} \leq 0$. When does $\frac{dL}{dt} = 0$? [6]

(iii) Using the results of part (b)(ii), or otherwise, show that the system (4) has a unique limit cycle. Is the limit cycle stable or unstable? *Give reasons for your answer.* [6]

(iv) Using the results of part (b)(i-iii), or otherwise, sketch the phase portrait of the system (4). [4]

Let $E = 1-3x^2-y^2$, for convenience

$\dot{x} = x(1-3x^2-y^2) - y(1+x)$, $\dot{y} = y(1-3x^2-y^2) + 3x(1+x)$

$\dot{x} = xE - y(1+x)$ $\qquad$ $\dot{y} = yE + 3x(1+x)$

FPs $\dot{x} = 0$, $\dot{y} = 0$ $\qquad$ $\underline{3x\dot{x} + y\dot{y}} = 3x^2(1-3x^2-y^2) + y^2(1-3x^2-y^2)$

$\cancel{+3xy}(1-3x^2-y^2)\ \cancel{-3xy}(1+x)$

$= \boxed{(3x^2+y^2)(1-3x^2-y^2) = 0}$

$\therefore\ 3x^2+y^2 = 0 \Rightarrow x = y = 0$ and $1-3x^2-y^3 = 0$

But $\dot{x} = \dot{y} = 0$ also implies $1-3x^2-y^2 = 0$ and $\therefore \begin{cases} -y(1+x) = 0 \\ x(1+x) = 0 \end{cases}$

But $x = -1$ does not satify $1-3x^2-y^2 = 0$ $\forall$ any y.

$\therefore$ only fixed pt $= \underline{0}$.

Now consider $L = (1-3x^2-y^2)^2 = E^2$, say, if $E \overset{\text{def}}{=} 1-3x^2-y^2$

Then $\dot{L} = 2E\dot{E} = 2E(-6x\dot{x} - 2y\dot{y})$

$= -2E(-6x^2E + \cancel{6xy(1+x)} - 2y^2E - \cancel{6xy(1+x)})$

$= -4E^2(3x^2+y^2) = -4E^2(1-E).$

$\therefore \quad \dot{L} = -4E^2(3x^2+y^2) = -4E^2(1-E)^*$ $\qquad 3x^2+y^2 = 1-E$

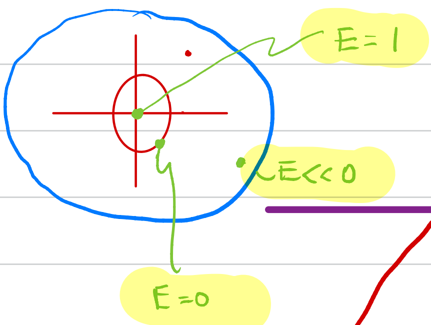
So $\dot{L} < 0$ except for when $E=0$ or $(3x^2+y^2)=0$

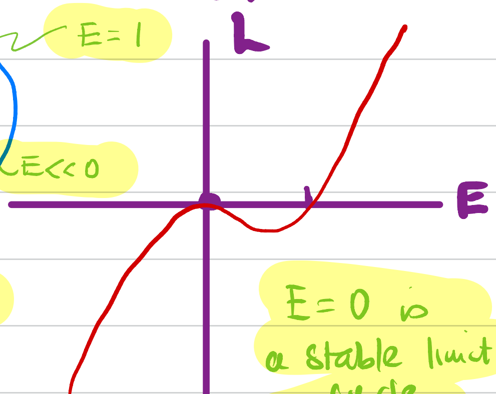
Note: $3x^2+y^2=0 \Leftrightarrow (x,y=0)$

$\therefore \dot{L} < 0$ at pts other than $(x,y)=0$

and (x,y) satisfying $3x^2+y^2=1$ (i.e. $E=0$)

Note $3x^2+y^2=1$ is an ellipse and $\begin{cases} \text{inside } 3x^2+y^2<1 \\ \text{outside } 3x^2+y^2>1 \end{cases}$





E=0 is a stable limit cycle

$\therefore \dot{L} < 0$, for $0 < E < 1$

$\dot{L} > 0$, for $1 < E$ (BUT imaginary (x,y)!)

$\dot{L} < 0$ $\qquad E < 0$

$E=1$: $(x,y)=0$ $\searrow$ as $t \nearrow$

$E=0$: $(x,y) \in$ ellipse

$E<0$: $(x,y) \nearrow$

T11.4

$\therefore \dot{L} < 0$ for $0 < E < 1$ & $E < 0$

i.e. L decreases as t increases to $L = 0$.

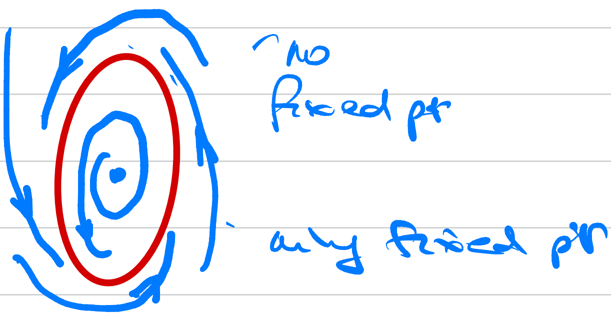
that is: $3x^2 + y^2 = 1$

Also

$L = 0$ is invariant as $\dot{L} = 0$ for $E = 0$ ($L = 0$).

$\therefore$ Orbits spiral to a periodic orbit given by $L = 0$

i.e. $3x^2 + y^2 = 1$



LINEARISATION AT THE ORIGIN

Eigenvalues of $\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$ are

$$\lambda_1, \lambda_2 = \frac{2 \pm \sqrt{4-16}}{2}$$

$$= 1 \pm 2\sqrt{3}\, i \quad \text{(unstable spiral)}$$

Linearisation at the fixed pt $x, y = (0,0)$:

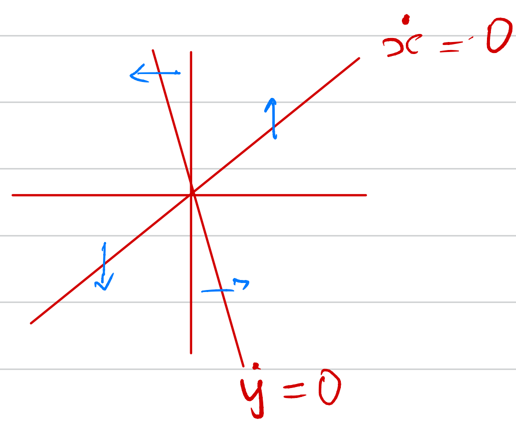
$$\dot{x} = x - y$$
$$\dot{y} = 3x + y$$

Hartman-Grobman give qualitative type — unstable spiral

Note $\dot{y} > 0$ for $3x + y > 0$

anticlockwise flow at the origin

null-cline analysis



16:09 4G 65

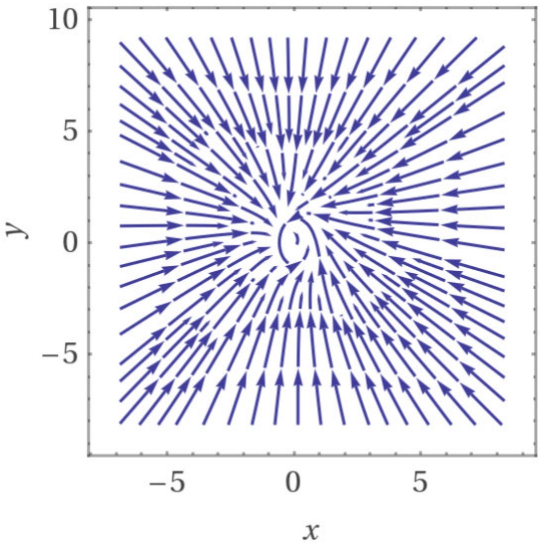
WolframAlpha

streamplot{x(1-3x^2-y^2)-y(1+x),y(1...

Input interpretation

stream plot	$\left(-(1+x)\,y + x\left(1-3\,x^2-y^2\right),\ 3\,x\,(1+x) + y\left(1-3\,x^2-y^2\right)\right)$

Plot

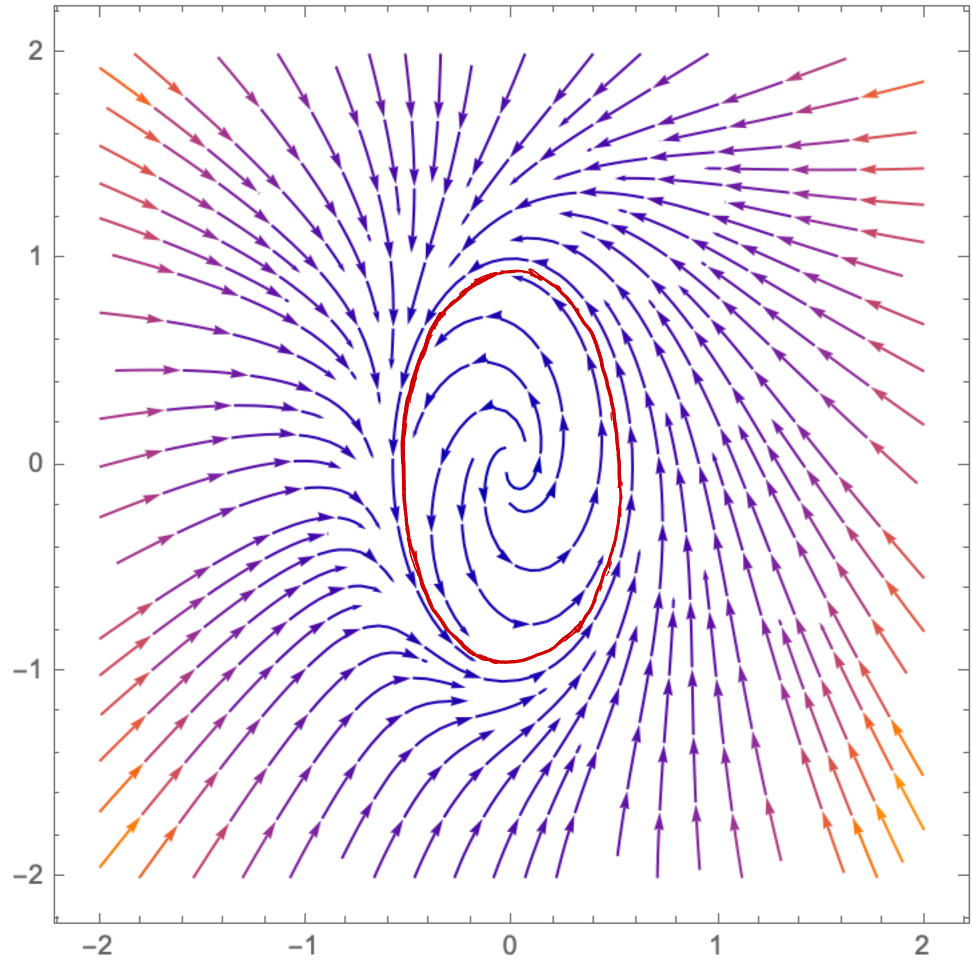


Not clear!

T11.6

```
StreamPlot[{x (1 - 3 x^2 - y^2) - y (1 + x),
  y (1 - 3 x^2 - y^2) + 3 x (1 + x)}, {x, -2, 2}, {y, -
```

]=



Thu.7

Question 3. [40 marks] Consider the second-order differential equation on $\mathbb{R}$

$$\ddot{x} + \dot{x}(Q(x,\dot{x}) - \mu) + 2x = 0 \qquad Q(x,\dot{x}) = 2x^2 + \dot{x}^2 \tag{3}$$

where μ is a real parameter.

(a) Rewrite (3) as a vector field on the plane: $\dot{z} = v(z), z = (x,y) \in \mathbb{R}^2$. [2]

(b) Show that this system has a unique fixed point and classify it for all non-zero parameter values (including bifurcation values). [8]

(c) Derive an equation for $\dot{Q} = \mathrm{d}Q/\mathrm{d}t$. [6]

(d) Let $\mu \leqslant 0$. Show that Q is a Lyapounov function for $\dot{z} = v(z)$. Hence prove that all orbits converge to the origin. [8]

(e) Let $\mu > 0$. Construct a trapping region for the flow; from it deduce the existence of at least one limit cycle. [8]

(f) Show that there is just one cycle, and write its equation. Verify that the vector field $v(z)$ is tangent to the cycle. [8]

T11.8

$$\ddot{x} + \dot{x}\left(Q(x,\dot{x}) - \mu\right) + 2x = 0$$

Let $\dot{x} = y$

$$Q(x,\dot{x}) = 2x^2 + \dot{x}^2 \qquad Q(x,y) = 2x^2 + y^2$$

$\dot{x} = y$

$Q(x,y) = 2x^2 + y^2$

(a) $\dot{x} = y, \quad \dot{y} = \ddot{x} = -y\left(Q(x,y) - \mu\right) - 2x$

(b) FPs $\dot{x} = \dot{y} = 0$: $y = 0$, $\underbrace{-\cancel{y}}_{=0}\left(Q(x,y) - \mu\right) - \underbrace{2x}_{\Rightarrow x=0} = 0$

Linearisation

$\dot{x} = y$

$\dot{y} = \mu y - 2x$

$$\begin{bmatrix} 0 & 1 \\ -2 & \mu \end{bmatrix}$$

(c) $\dot{Q} = \dfrac{dQ}{dt} = \dfrac{\partial Q}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial Q}{\partial y} \cdot \dfrac{dy}{dt}$

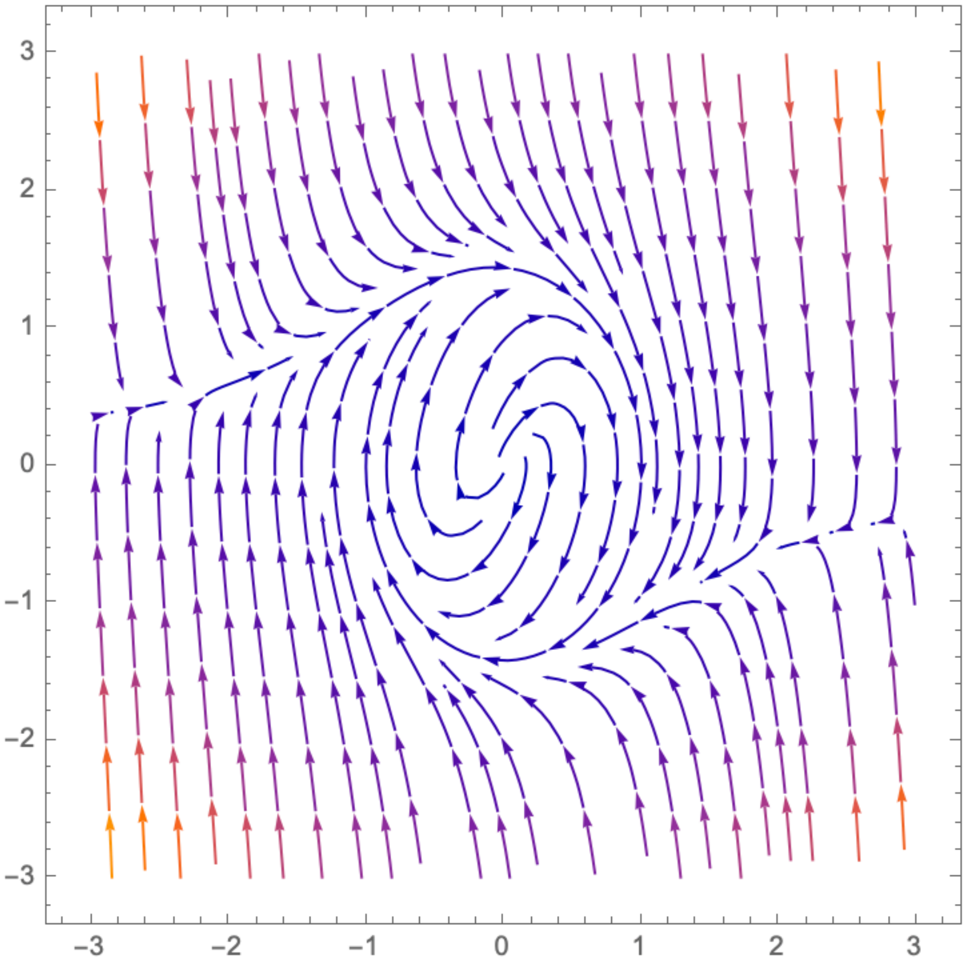
$$= \cancel{4xy} - 2y^2\left(Q(x,y) - \mu\right) - \cancel{4xy}$$

$$= -2y^2 Q(x,y) + 2y^2\mu$$

For $\mu \leq 0$, $\dot{Q} \leq 0$

```
mu = 2;
StreamPlot[{y, -y (2 x ^ 2 + y ^ 2 - mu) - 2 x},
 {x, -3, 3}, {y, -3, 3}]
```

T11.9



(d) $\mu \leq 0$, Then $\dot{Q}(x,y) = -2y^2 Q(x,y) - 2y^2(-\mu)$

$Q(x,y) \geq 0$

If $-\mu \geq 0$

$$\left.\begin{array}{l} Q(x,y) \text{ is PD} \\ \dot{Q}(x,y) \text{ is non-positive} \end{array}\right\} Q \text{ is a LF.}$$

LINEARISATION

T11.11

Eigenvalues $\lambda(\lambda-\mu)+2=0$

$$\lambda = \frac{\mu \pm \sqrt{\mu^2-8}}{2}$$

Note $\forall \mu>0$, eigenvalues have positive real parts

$\therefore$ either unstable spiral or node at the origin

by Hartman - Grobman Theorem

Now

$$\dot{Q} = -2y^2\left(Q(x,y)-\mu\right)$$

Consider an ellipse given by

$$Q(x,y) = 2x^2+y^2 = C > \mu > 0.$$

Then $\dot{Q}(x,y) \leq 0$ on C.

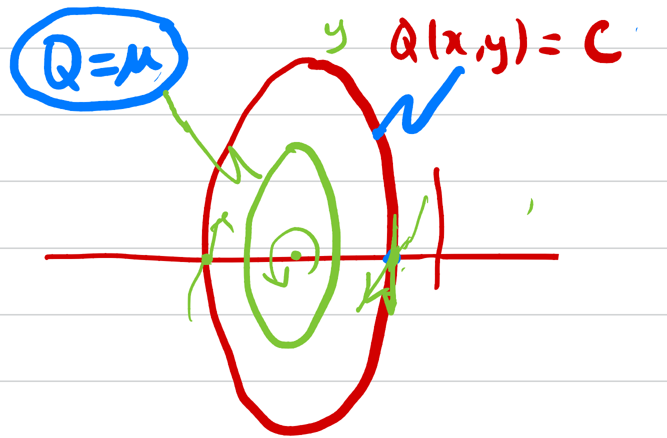
and

$\dot{Q}(x,y) < 0$ for $y \neq 0$.

Finally when $y=0$ on $Q(x,y)=C$

$$\dot{x}=y, \quad \dot{y}=-y\left(Q(x,y)-\mu\right)-2x$$

i.e. $\dot{x}=0$, $\dot{y}=-2x$, $x\neq 0$



$\dot{Q}=0$ for $Q=\mu$

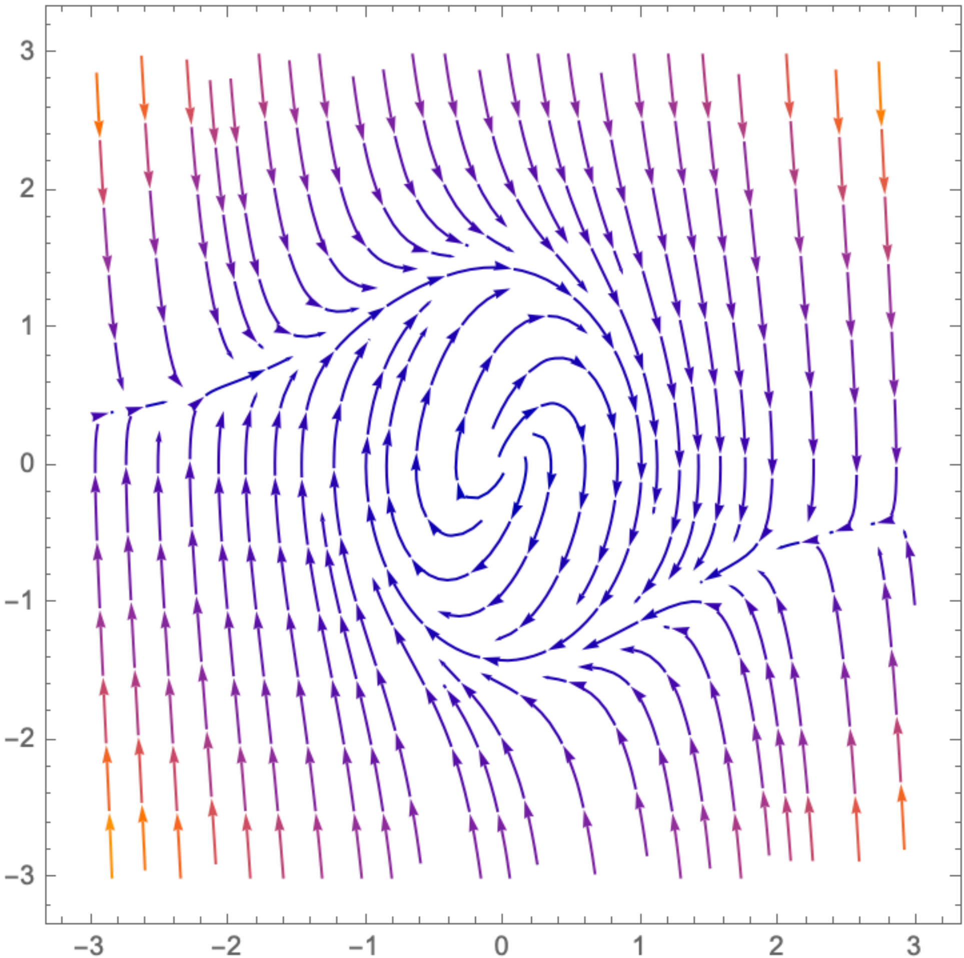
$\therefore$ $Q=\mu$ is invariant curve of the system and fixed pt free $\therefore$ periodic orbit

* Note $Q(x,y)=\mu \Rightarrow \dot{x}=y$, $\dot{y}=-2x$

$$\Rightarrow \frac{dx}{y}=\frac{dy}{-2x} \qquad x^2+\frac{y^2}{2}=\text{const}$$

T11.13

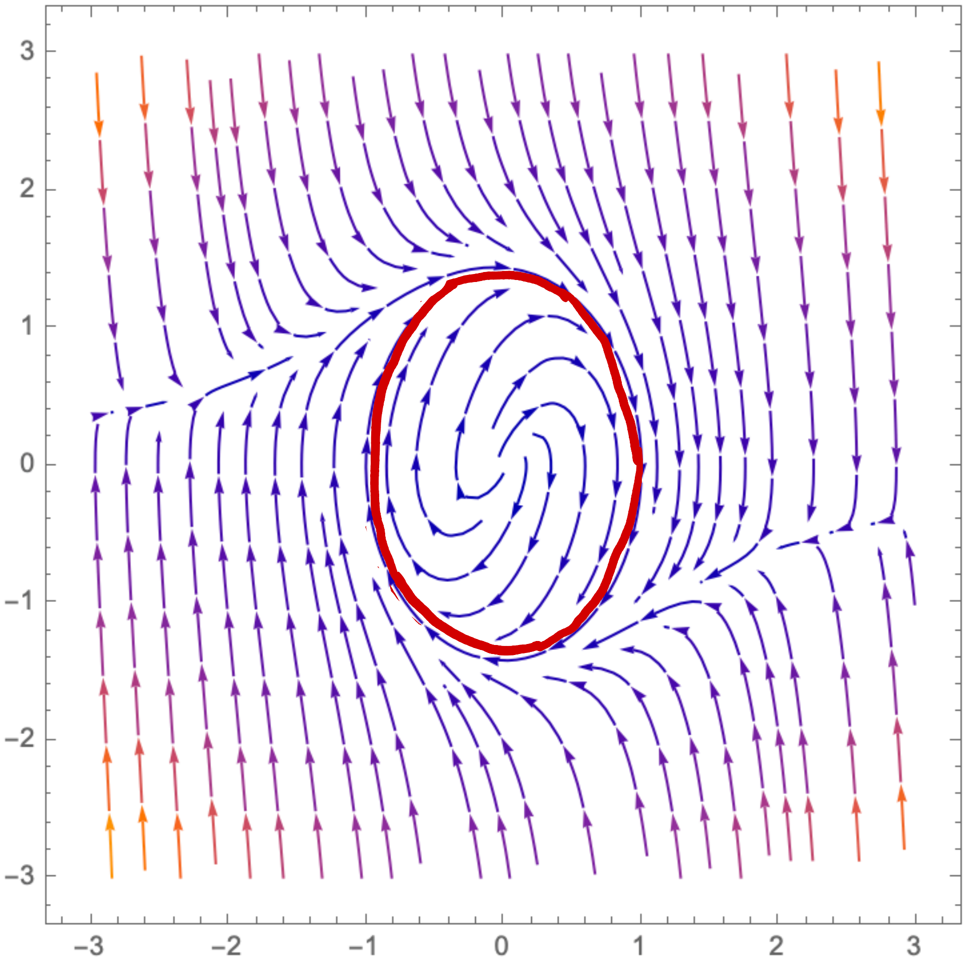
```
mu = 2;
StreamPlot[{y, -y (2 x^2 + y^2 - mu) - 2 x},
  {x, -3, 3}, {y, -3, 3}]
```



T11. 14

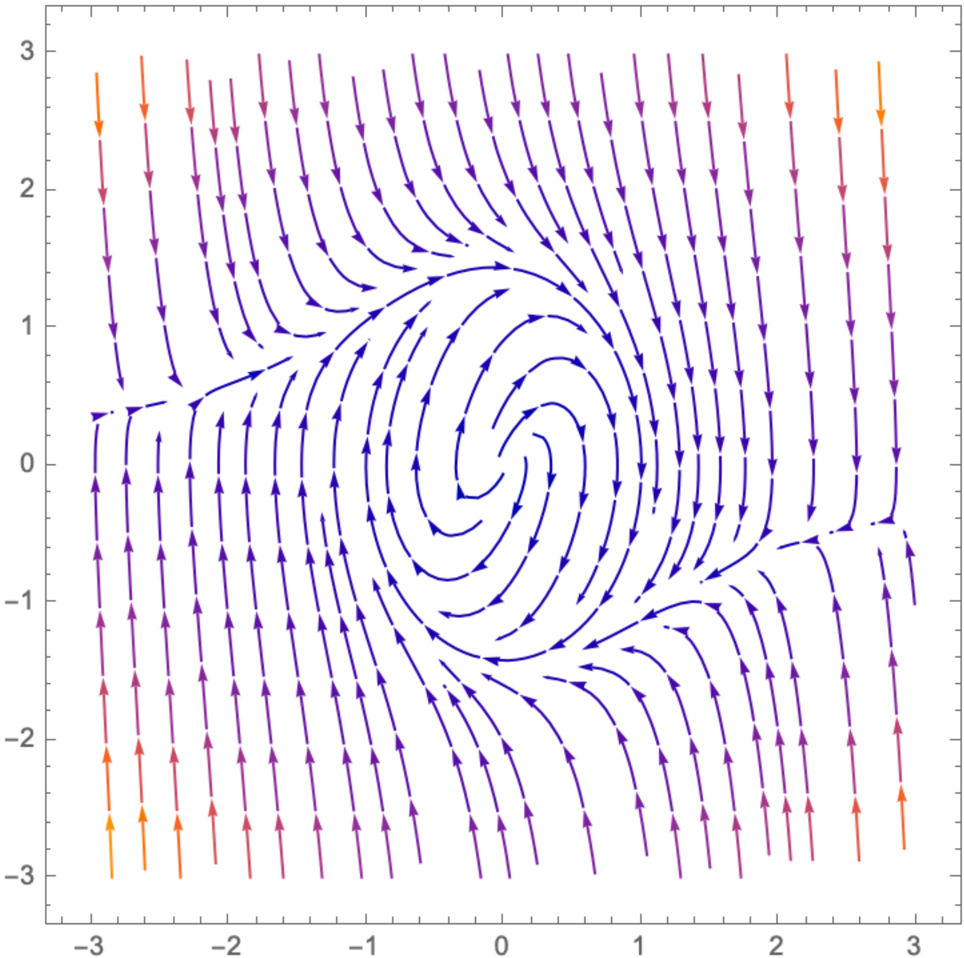
```
mu = 2;
StreamPlot[{y, -y (2 x^2 + y^2 - mu) - 2 x},
  {x, -3, 3}, {y, -3, 3}]
```

"Location" of limit cycle



T11.15

```
mu = 2;
StreamPlot[{y, -y (2 x^2 + y^2 - mu) - 2 x},
 {x, -3, 3}, {y, -3, 3}]
```



Question 3. [44 marks] (Two-dimensional systems)

(a) Classify the fixed point $\boldsymbol{x}^* = (0,0)$ for each of the following two-dimensional linear systems and state, with justification, whether $\boldsymbol{x}^*$ is either attracting, Liapunov stable, asymptotically stable, or neither.

(i) $\dot{x} = x + 3y,\ \dot{y} = 1 + 2y.$ **[4]**

(ii) $\dot{x} = 4x + y,\ \dot{y} = -3x.$ **[4]**

(iii) $\dot{x} = -2x,\ \dot{y} = x - 2y.$ **[4]**

(b) Consider the following two-dimensional system

$$\dot{x} = x^3 - x\,, \quad \dot{y} = y + 1 - e^x\,.$$

(i) Identify the fixed points and classify the type of each fixed point by performing a linear stability analysis. **[6]**

(ii) Find equations for all the nullclines; sketch these in the phase plane, indicating the direction of motion along each nullcline. **[6]**

(iii) Using Parts (i) and (ii) sketch the entire phase portrait, indicating typical trajectories and the direction of motion along these trajectories. **[6]**

(c) Consider the following conservative system

$$\ddot{x} = rx - e^x =: -\frac{dV(x)}{dx}, \quad r > 0\,. \tag{1}$$

(i) Find an expression for $V(x)$ and show that for $0 < r < e$ the potential $V(x)$ has no stationary points and for $r > e$ the potential $V(x)$ has two stationary points. **[4]**

(ii) Sketch the graph of $V(x)$ when $0 < r < e$ and $r > e$. **[3]**

(iii) Perform a transformation to turn the second-order equation in (1) into a system of two coupled first-order differential equations and find the associated conserved quantity. **[2]**

(iv) Using Parts (i), (ii) and (iii) sketch the phase portrait when $0 < r < e$ and $r > e$, indicating typical trajectories, the direction of motion along these trajectories and, where appropriate, any homoclinic orbits. **[5]**

WEEK 12. Exam questions

Exam 2018 (3b)

Investigate: $\dot{x} = x^3 - x$

$\dot{y} = y + 1 - e^x$

Fixed point set

$\dot{x} = \dot{y} = 0$:

$x^3 - x = 0$, $y + 1 - e^x = 0$.

$x(x-1)(x+1) = 0$

$x = 0$, $y = 0$

$x = +1$, $y = e - 1 > 0$

$x = -1$, $y = e^{-1} - 1 < 0$.

Jacobian $J = \begin{bmatrix} 3x^2 - 1 & 0 \\ -e^x & 1 \end{bmatrix}$

At $x = (0,0)$ $J = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$ - saddle with $\lambda_1 = -1$ and $\lambda_2 = 1$

Eigenvalues - Eigen vectors

$\lambda_1 = -1$: $\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} x \\ y \end{bmatrix}$: $\underline{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda_2 = +1$ $\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = +1\begin{bmatrix} x \\ y \end{bmatrix}$ $\underline{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\underline{x} = (1, e-1)$

$J = \begin{bmatrix} 2 & 0 \\ -e & 1 \end{bmatrix}$ $\lambda_1 = 2$, $\lambda_2 = 1$

$\begin{bmatrix} 2 & 0 \\ -e & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ $y = -ex$
$\approx -3x$

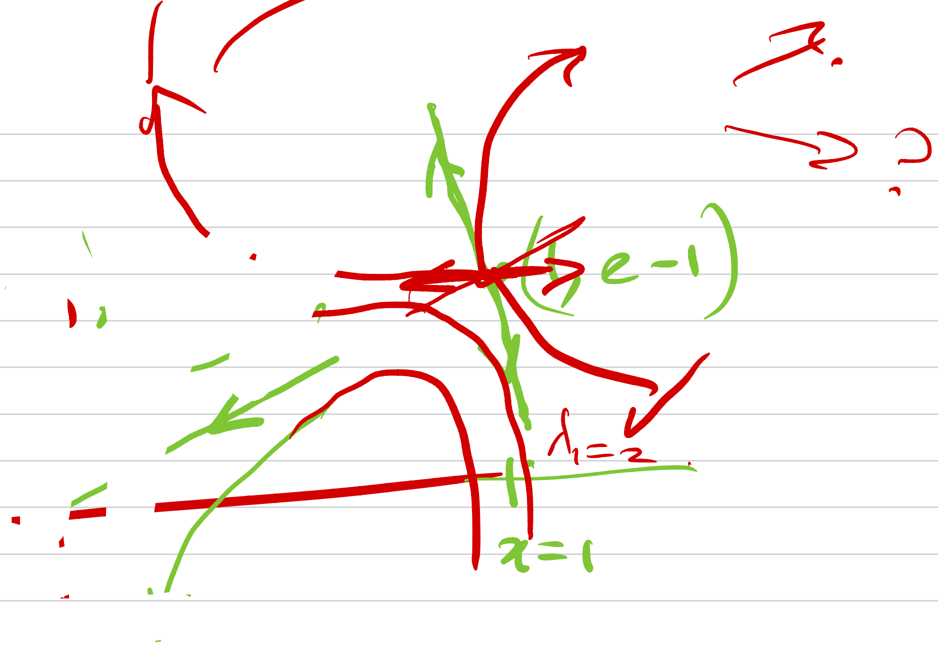
$\underline{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x=-1$, $y=e^{-1}-1$

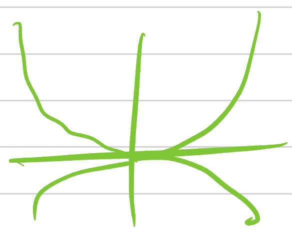
$$J=\begin{bmatrix}2 & 0\\ -e^{-1} & 1\end{bmatrix} \qquad \lambda_1=2,\quad \lambda_2=1$$

$y=-e^{-1}x$, $x=-3y$ $(\lambda_1=2)$ $v_1=\begin{bmatrix}-3\\1\end{bmatrix}$

$x=0$, $v_2=\begin{bmatrix}0\\1\end{bmatrix}$. $(\lambda_2=1)$



$$\begin{bmatrix}3x^2-1 & 0\\ -e^{x} & 1\end{bmatrix}$$



$$\begin{bmatrix}2 & 0\\ -e & 1\end{bmatrix}$$

$\lambda_1=2$, $\lambda_2=1$

$$\begin{bmatrix}2 & 0\\ -e & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=1\begin{bmatrix}x\\y\end{bmatrix}$$

$\lambda_1=2$: $-ex+y=2y$

$-ex=y$

$\lambda_2=1$ $\quad -ex=y$

On $x=0$

$\dot y = y$

On $x=+1$

fixed x

$\dot y = y + (1 - e^{x})$

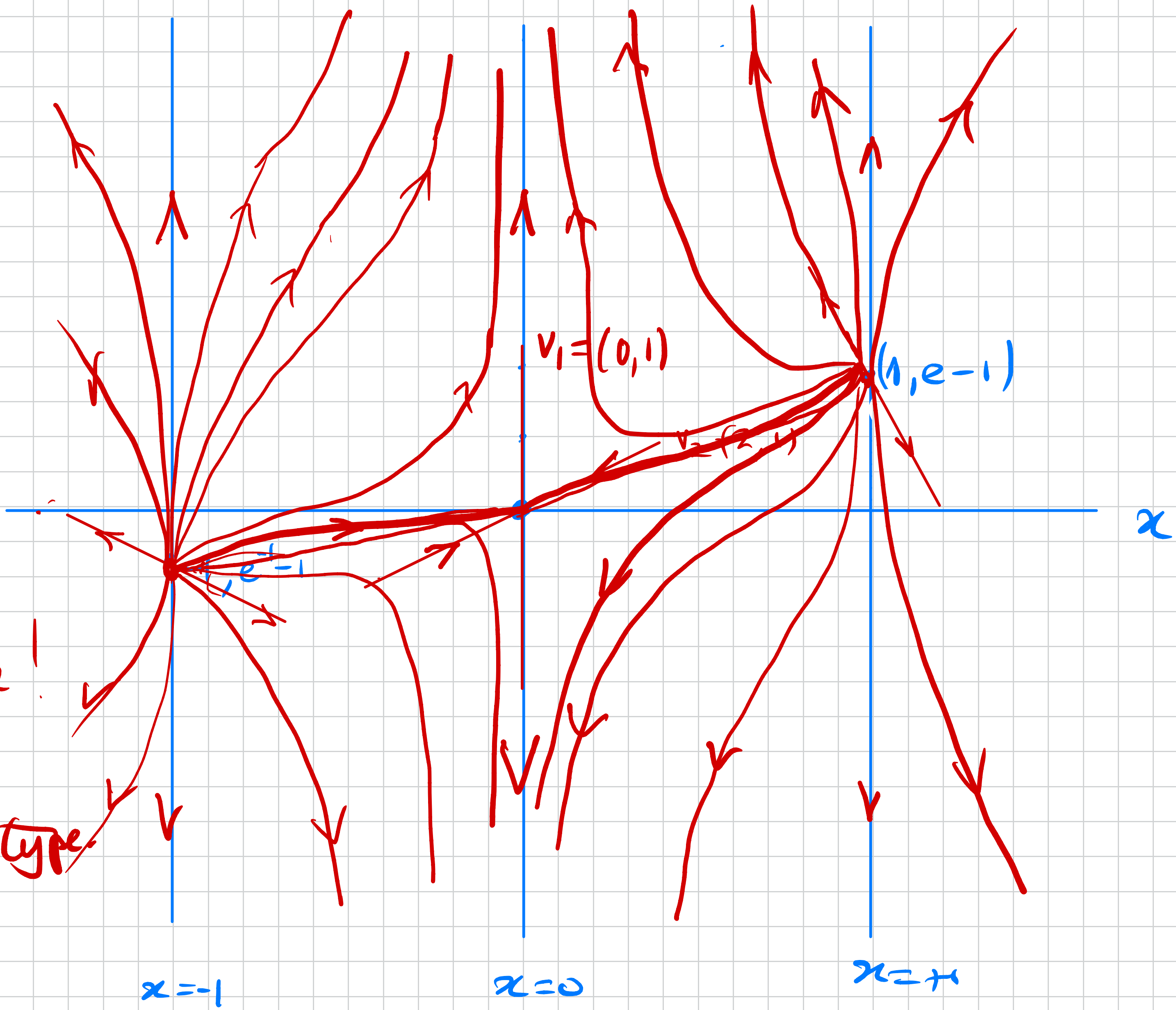
$y \gg 0$, $\dot y \gg 0$ etc

hence

This is a tough one!

2018!

I have got the correct qualitative type.



3 $\ddot{x} = rx - e^x \Rightarrow \dot{x} = y, \; \dot{y} = rx - e^x$

FPs e^x $y = 0$, $rx - e^x = 0$

For tangency $rx = e^x = 0$

and $\frac{\partial}{\partial x}(rx - e^x) = 0$.

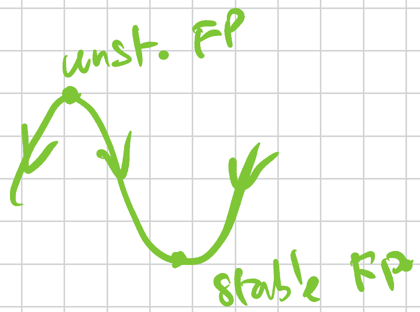
$r - e^x = 0$ $\quad x \Rightarrow \ln(r)$

$r\ln(r) - e^{\ln(r)} = 0$

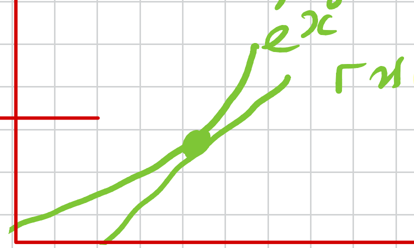
i.e. $r\ln(r) - r$

$r(\ln(r) - 1) = 0$

$r = 0$, $\ln(r) = 1 \Rightarrow r = e$.



$\therefore$ $r = e$, gives tangency for $y = e^x$ and $y = rx$



So as r increases from $r = 0$ (graph $y = rx$ is an horizontal line) to $r = e$ where graphs $y = rx$ & e^x are tangent to $r > e$ where graphs intersect in 2 points

$$\dot{x} = x^3 - x \qquad \dot{y} = y + 1 - e^{+x}$$

FPs

$$\cancel{x^3 - x = 0} \qquad y + 1 - e^{+x} = 0$$

$$x(x^2 - 1) = 0$$

$$x = 0,\ x = 1,\ x = -1$$

$$x = 0,\ y = 0$$

$$x = 1 \quad y = e - 1 > 0$$

$$x = -1 \quad y = e^{-1} - 1 < 0$$

Nullclines

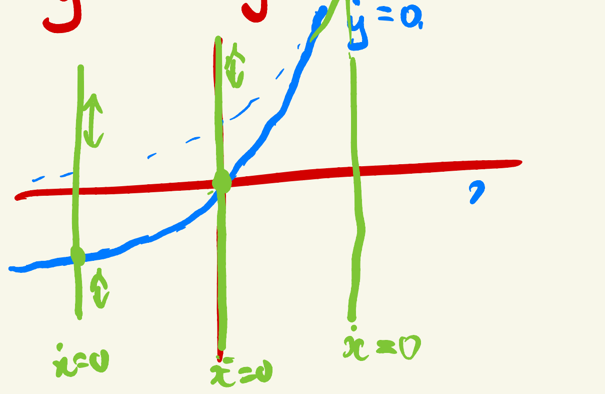
$$x \equiv 0,\ x \equiv 1,\ x \equiv -1$$

$$\dot{x} = 0 \qquad \checkmark$$

$$x(t),\ x(0) = 0$$

$$x(t) \equiv 0$$

$$\dot{y} = 0 \qquad y = e^x - 1$$



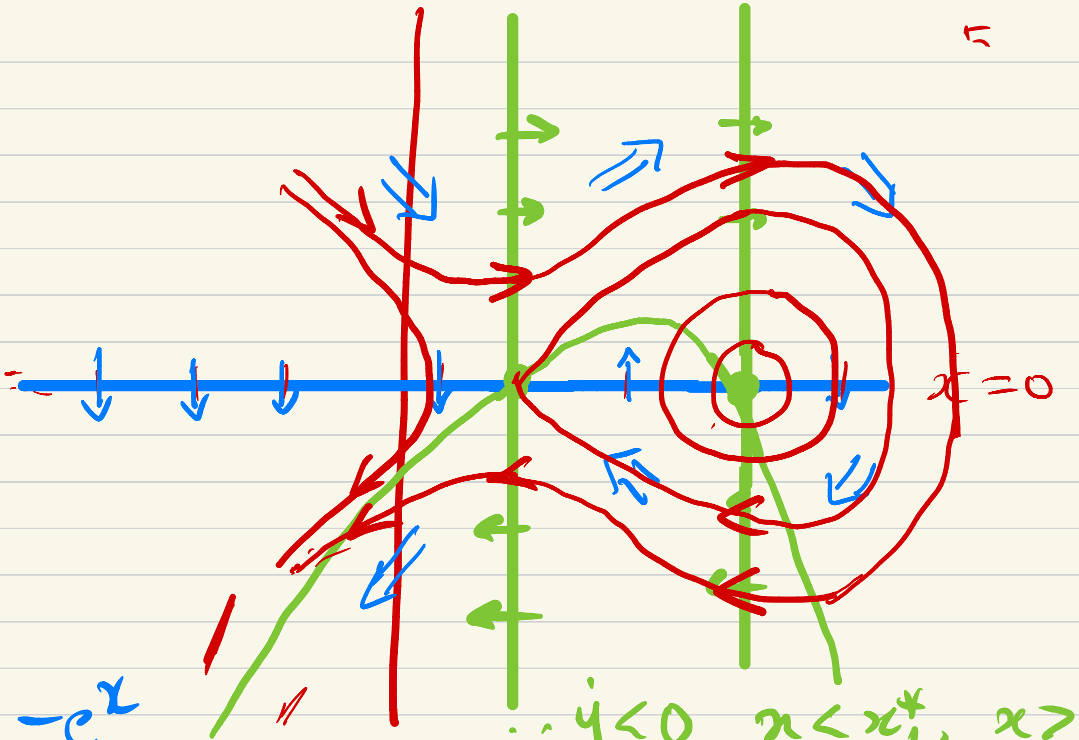
$\dot{x} = y$

$rx - e^x$

$\dot{y} = rx - e^x$

$\dot{x} = 0 \qquad y = 0$

$\dot{y} = 0.$



Graph of $f(x) = rx - e^x$

$\dot{y} < 0 \quad x < x_1^*, \; x > x_2^*$

$\dot{y} > 0 \quad x_1^* < x < x_2^*$

For **r > e**, there exist two fixed points x_1^*, x_2^*

$0 < x_1^* < e < x_2^*.$

For $x \to -\infty \quad f(x) \approx rx \quad (e^{-x} \to 0)$ "/"

$x \to +\infty \quad f(x) \approx -e^x$ "\"

It is a conservative system

$E = \frac{y^2}{2} + V(x)$

So saddles and centres

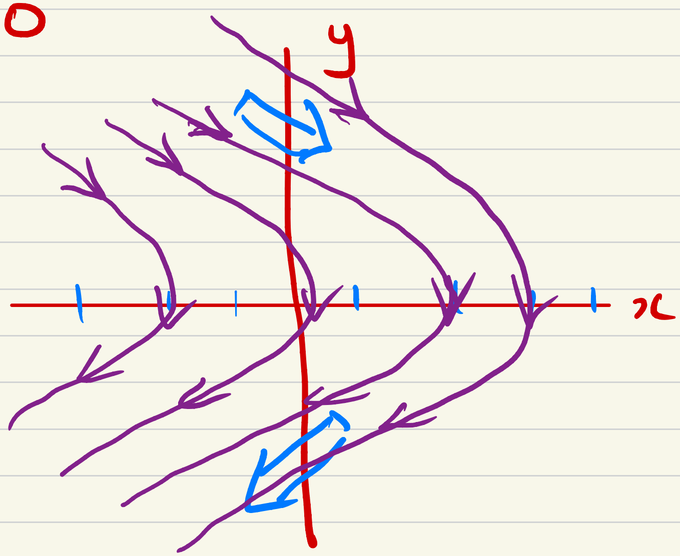
What about phase portrait for $0 < r < e$

$\rightarrow$ There are no fixed points, as you decrease r, the fixed points get closer together and disappear for $r < e$.

$\dot{x} = y$, $\begin{cases} \dot{x} > 0 & \text{in upper half plane } (y > 0) \\ \dot{x} < 0 & \text{in lower half plane } (y < 0) \end{cases}$

$y = 0$ is null cline (flow vertical as $\dot{x} = 0$ for $y = 0$)

For all x, $\dot{y} < 0$



Question 2. [34 marks] Bifurcations

Consider the differential equation

$$\dot{x} = rx - x^2 + x^4 \quad ,$$

which depends on a real parameter r.

a) Compute the range of parameter values for which the trivial fixed point $x_* = 0$ is linearly stable and the range of parameter values for which it is linearly unstable. If you find bifurcations of the trivial fixed point, compute the parameter value of the bifurcation point and state with a reason the type of the bifurcation. **[8]**

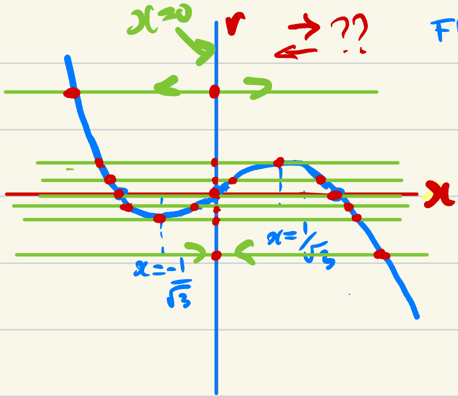
b) Compute the parameter values $r = r_*$ and the points $x = x_*$ in phase space where nontrivial fixed points, $x_* \neq 0$, undergo a saddle node bifurcation. **[8]**

c) For each saddle node bifurcation computed in part b) decide whether the pair of fixed points is generated for $r < r_*$ or $r > r_*$. **[10]**

d) Using the results from parts a) - c), or otherwise, sketch the bifurcation diagram of the differential equation. Your diagram should indicate the stability of each fixed point. The diagram should also contain phase portraits for parameter values where phase portraits qualitatively differ. **[8]**

$$\dot{x} = rx - x^2 + x^4 = x(r - x + x^3) = f(x)$$

FPs $\quad x=0 \qquad r = x - x^3 \qquad \frac{dr}{dx} = 1-3x^2$



FP set: y-axis & cubic

3 bifurcation pts

$x=0$ (transcritical?)

$x=1/\sqrt{3}$ (saddle-node?) sub-critical?

$x=-1/\sqrt{3}$ (saddle-node?) super critical

Check with Taylor coefficients? Etc.

Linear stability $\quad f'(x) = r - 2x + 4x^3$

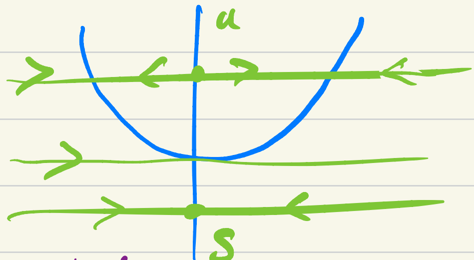
$f'(0) = r, \quad r>0$ (u) $\quad r<0$ (s)

Just comments!

$$\dot{x} = r - x^4$$

looks like a saddle-node bifurcation. but isn't

$$\dot{x} = rx - x^3$$



That's it folks, that's all for now.

Thanks for the feedback, I did try my best in getting the material over. I accept your concern over organisation. Part of that was out of my control with a bad year for QMplus. - A general view on the School.

Please get in touch for help

Question 3. [36 marks] Two-dimensional systems

Consider the system of two differential equations

$$\dot{x} = y \quad , \quad \dot{y} = -x + x^3 - y \quad .$$

a) Compute the fixed points of this two dimensional dynamical system. For each fixed point perform a linear stability analysis and classify the type of fixed point. **[6]**

b) Is this system conservative? Is it a gradient system? Is it reversible? State reasons and compute, if appropriate, the potential. **[12]**

c) Sketch the flow in the phase plane in a small neighbourhood of each fixed point. **[4]**

d) Construct a nullcline diagram for this system as follows:

(i) Compute the nullclines of the system of differential equations.

(ii) Sketch the nullclines in the phase plane.

(iii) The nullclines partition the phase plane into different regions. For each region, and on each nullcline, indicate the direction of the flow. **[8]**

e) Using the results from parts c) - d), or otherwise, sketch the full phase portrait of the two dimensional system. The phase portrait should be consistent with the diagram produced in part d). If the system has a stable fixed point then shade its basin of attraction. **[6]**