Queen Mary
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MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 2 (Real Numbers)

1. Consider the following sets:

   (a) $A = [-1, 3]$.
   (b) $B = (-1, 3)$.
   (c) $C = (-1, 3) \cap [-3, 1]$.
   (d) $D = (1, 2) \cup [7, 8]$.
   (e) $E = \{z \in \mathbb{R} : z^3 < 2\}$.
   (f) $F = \{n^2 : n \in \mathbb{N}\}$.
   (g) $G = \{z \in \mathbb{R} : 0 < z^2 < 1\}$.

   For each of (a)-(g), answer the following questions (fully justify your answers):

   (i) Does this set have an upper bound?
   (ii) Does this set have a supremum?
   (iii) Does this set have a maximum?
   (iv) Does this set have a lower bound?
   (v) Does this set have an infimum?
   (vi) Does this set have a minimum?

   Solutions. I will provide solutions for (a),(b),(e),(f).

   (a) (i) This set has an upper bound, namely $x = 3$. Indeed, if $y \in A$ then $y \leq 3$ which shows that 3 is an upper bound. Of course, any $x' \geq 3$ will also be an upper bound for A.

   (ii) This set has a supremum, namely $x = 3$. Indeed, we have already seen that x is an upper bound for A. To show that x is a least upper bound, let $z < x$. If $z \in A$ then $y = (z+x)/2$ is such that $y \in A$ and $y > z$, which shows that z is not an upper bound for A. If $z \notin A$ (i.e. if $z < -1$) then simply taking $y = -1$ achieves the same result.

   (iii) We have $\sup(A) = 3 \in A$ and so by Lemma 2.11 we see that A has a maximum, namely $\max(A) = 3$.

   (iv) This set has a lower bound, namely $x = -1$. Indeed, if $y \in A$ then $y \geq -1$ which shows that -1 is a lower bound. Of course, any $x' \leq -1$ will also be a lower bound for A.

(v) This set has an infimum, namely $x = -1$. Indeed, we have already seen that x is a lower bound for A. To show that x is a greatest lower bound, let $z > x$. If $z \in A$ then $y = (z + x)/2$ is such that $y \in A$ and $y < z$, which shows that z is not a lower bound for A. If $z \notin A$ (i.e. if $z > 3$) then simply taking $y = 3$ achieves the same result.

(vi) We have $\inf(A) = -1 \in A$ and so by Lemma 2.11 we see that A has a maximum, namely $\min(A) = -1$.

(b) (i) This set has an upper bound, namely $x = 3$ (and anything greater than this). The proof is similar to (a)(i) above.

(ii) This set has a supremum, namely $x = 3$. The proof is similar to (a)(ii) above.

(iii) This set does *not* have a maximum. Indeed we have $\sup(B) = 3$ but $3 \notin B$ and hence, by Lemma 2.11, B does not have a maximum.

(iv) This set has a lower bound, namely $x = -1$ (and anything less than this). The proof is similar to (a)(iv) above.

(v) This set has an infimum, namely $x = -1$. The proof is similar to (a)(v) above.

(vi) This set does not have a minimum. Indeed we have $\inf(B) = -1$ but $-1 \notin B$ and hence, by Lemma 2.11, B does not have a maximum.

(e) If $z < 0$ then $z^3 < 0 < 2$. On the other hand if $z \geq 0$ then $z^3 < 2$ if and only if $z < \sqrt[3]{2}$. We conclude that

$$E = \{z \in \mathbb{R} : z < \sqrt[3]{2}\}.$$

(i) This set has an upper bound, namely $x = \sqrt[3]{2}$ (and any number greater than this). This is clear from the above description of E.

(ii) This set has a supremum, namely $x = \sqrt[3]{2}$. Indeed, we have already seen that x is an upper bound for E. To show it is the least upper bound, suppose that $z \in \mathbb{R}$ with $z < x$. Then choose $y = (z + x)/2$. We can show (check this yourself!) that $z < y$ and that $y \in E$. Therefore z is not an upper bound for E, and we conclude that x is the least upper bound.

(iii) This set does not have a maximum, by Lemma 2.11: $\sup(E) = \sqrt[3]{2} \notin E$.

(iv) This set does not have a lower bound! Indeed, suppose for a contradiction that $x \in \mathbb{R}$ is a lower bound for E. This means that for all $y \in E$ we have $x \leq y$. Choose $y \in E$. This means that $y < \sqrt[3]{2}$. Since $x \leq y < \sqrt[3]{2}$ we have $x \in E$ also. But then $x - 1 < x < \sqrt[3]{2} \in E$ which contradicts the assumption that x is a lower bound for E.

(v) Since E does not have a lower bound, it does not have an infimum (which, when it is defined, is the greatest of all the lower bounds; if there are no lower bounds then there is no infimum).

(vi) Since E does not have an infimum, it does not have a minimum (since by Lemma 2.11 the existence of a minimum implies the existence of an infimum).

(f) We can rewrite G as:

$$G = \{z \in \mathbb{R} : -1 < z < 1 \text{ and } z \neq 0\} = (-1, 1) \setminus \{0\}.$$

The point is that removing $\{0\}$ from $(-1, 1)$ does not affect the upper bound, supremum, etc., since it happens in the "middle" of the set. Thus, the following arguments are similar to (b), and I will not repeat them in full.

(i) This set has an upper bound, namely $x = 1$ and anything greater.
(ii) This set has a supremum, namely $x = 1$.
(iii) This set does not have a maximum because $1 \notin G$.
(iv) This set has a lower bound, namely $x = -1$ and anything less.
(v) This set has an infimum, namely $x = -1$.
(vi) This set does not have a minimum because $-1 \notin G$.

2. Let $A = \{1/n : n \in \mathbb{N}\}$.

(a) Find, with brief justification, a lower bound for A.
(b) Suppose $x \in \mathbb{R}$ with $x > 0$. Is x a lower bound for A? Justify your answer. (You may use any theorems from the course providing you clearly state which theorem you are using.)
(c) Does A have an infimum? Prove your answer.

Solution We have $A = \{\frac{1}{n} \ : \ n \in \mathbb{N}\}$.

(a) 0 is a lower bound for A since all $x \in A$ satisfy $x > 0$. (*Of course, any negative number is also a lower bound.*)

(b) Any $x \in \mathbb{R}$ with $x > 0$ is not a lower bound since by Corollary 2.16 there is an $n \in \mathbb{N}$ with $\frac{1}{n} < x$, and this $\frac{1}{n} \in A$. This means that we have found something in A which is smaller than x, so x is not a lower bound.

Alternative: $x > 0$ is a real number, and hence so is $\frac{1}{x}$. Therefore by Theorem 2.14 there is a natural number n with $n > \frac{1}{x}$, i.e. $\frac{1}{n} < x$. This $\frac{1}{n} \in A$ so x is not a lower bound. (*Recall: To show that x is not a lower bound we need to show there is something in A that is smaller than x.*)

(c) From our answer to (a) and (b) it follows immediately that 0 is the infimum, for we have shown that 0 is a lower bound and that nothing bigger than 0 is a lower bound. (*A correct answer to part (c) needs to contain two observations: First, that zero is a lower bound and secondly that nothing bigger than zero is a lower bound (i.e., part (b) of the question). You do not need to reprove part (b) – it was there for a reason!*)

3. Suppose $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ are sets, and that $a = \sup A$ and $b = \sup B$ both exist.

   (a) Prove that $A \cap B$ is bounded above by a and also by b. (This means that a and b are both upper bounds for $A \cap B$.)

   (b) Suppose $A \cap B \neq \emptyset$. Prove that $A \cap B$ has a supremum, m say, and that $m \leq \min\{a, b\}$.

   (c) Assuming $A \cap B \neq \emptyset$, is it necessarily the case that $m = \min\{a, b\}$? Either give a proof or give a counterexample.

   Something to think about (not part of the question). What happens in part (b) if $A \cap B = \emptyset$?

   Solution.

   (a) By definition of supremum, a is an upper bound for A. Let x be any element of $A \cap B$. In particular, x is an element of A, and hence $x \leq a$. Thus a is an upper bound for $A \cap B$. (If $A \cap B = \emptyset$ then a is vacuously an upper bound for $A \cap B$.)

   (b) We know from part (a) that for all $x \in A \cap B$, we have $x \leq a$ and $x \leq b$. It follows that $m = \min\{a, b\}$ is an upper bound for $A \cap B$. Since $A \cap B$ is non-empty, it has a supremum (by the completeness axiom). Since the supremum is the *least* upper bound, it must be less than or equal to any given upper bound such as m.

   (c) It is not always the case. E.g., suppose $A = \{0, 1\}$ and $B = [0, 1)$. Then $\sup(A) = \sup(B) = 1$. However, $\sup(A \cap B) = \sup(\{0\}) = 0 < \min\{1, 1\}$.

4. First, restudy our proof of Theorem 2.19 from the lecture notes (stating that there exists a number $x \in \mathbb{R}$ with $x^2 = 2$). Modify the proof of Theorem 2.19 to show that there is a real number x with $x^2 = 19$.

   Solution. Let $A = \{z : z^2 \leq 19\}$. Then $1 \in A$ (since $1^2 = 1 \leq 19$). Hence A is non-empty. Also if $z > 10$ then $z^2 > 100 \not\leq 19$; i.e., $z \notin A$. Therefore 10 is an upper bound for A: in particular A is bounded above. Hence, by the completeness axiom A has a supremum $x = \sup(A)$. Note $1 \leq x \leq 10$. We aim to show that $x^2 = 19$. Now, either $x^2 = 19$, $x^2 < 19$ or $x^2 > 19$, so we aim to show that the latter two cases cannot occur.

   <u>Suppose that $x^2 < 19$.</u> We try to get a contradiction by finding a number bigger than x which is in A. Let $\varepsilon = 19 - x^2$. Consider the number $x + 1/n$.

Then

$$\begin{aligned}(x+1/n)^2 &= x^2 + 2x/n + 1/n^2 \\ &= 19 - \varepsilon + 2x/n + 1/n^2 \\ &\le 19 - \varepsilon + 2x/n + 1/n && \text{since } 1/n^2 \le 1/n \\ &= 19 - \varepsilon + (2x+1)/n \\ &\le 19 - \varepsilon + 21/n && \text{since } x \le 10\end{aligned}$$

Hence, if we pick $n > 21/\varepsilon$ (which we can do by Theorem 2.14) we have $(x+1/n)^2 < 19$: that is $x + 1/n \in A$. Since $x + 1/n > x$ this contradicts the fact that x was *an* upper bound for A.

Now suppose $x^2 > 19$. We consider the number $(x - 1/n)$. Let $\varepsilon = x^2 - 19$. Then

$$\begin{aligned}(x-1/n)^2 &= x^2 - 2x/n + 1/n^2 \\ &= 19 + \varepsilon - 2x/n + 1/n^2 \\ &> 19 + \varepsilon - 2x/n && \text{since } 1/n^2 > 0 \\ &> 19 + \varepsilon - 20/n && \text{since } x \le 10\end{aligned}$$

Hence, if we pick $n > 20/\varepsilon$ we have $(x - 1/n)^2 > 19$. Since $1/n \le 1$ and $x \ge 1$ (since $1 \in A$) we see that $x - 1/n \ge 0$. Hence, if $y > x - 1/n$ we have $y^2 > (x - 1/n)^2 > 19$, so $y \notin A$: i.e., $x - 1/n$ is an upper bound for A. This contradicts the fact that x was the *least* upper bound.

Since neither of the latter two cases can occur we must have $x^2 = 19$.

5. **Challenge.** Let I_1, I_2, I_3, ... be a *decreasing sequence of nested closed intervals*, i.e.,

   - For all $n \in \mathbb{N}$, $I_n = [a_n, b_n]$ is a closed interval.
   - $\forall n \in \mathbb{N} : I_{n+1} \subseteq I_n$.
   - $\forall \varepsilon > 0\ \exists n \in \mathbb{N} : |I_n| < \varepsilon$, where $|I_n| = b_n - a_n$ is the length of the interval.

   Show, using the Completeness Axiom, that there exists exactly one $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N}\ :\ x \in I_n$ (this is known as the "nested interval principle").

   Solution. Let I_1, I_2, I_3, ... be a sequence of intervals with the following properties.

   - For all $n \in \mathbb{N}$, $I_n = [a_n, b_n]$ is a closed interval.
   - $\forall n \in \mathbb{N} : I_{n+1} \subset I_n$.

- $\forall \varepsilon > 0 \ \exists n \in \mathbb{N} : |I_n| < \varepsilon$, where $|I_n| = b_n - a_n$ is the length of the interval.

We first want to show that there exists some $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N} \ : \ x \in I_n$. Because $A := \{a_1, a_2, a_3, \ldots\}$ is bounded above (and non-empty), we can set $x := \sup(A)$. Because every b_n is an upper bound for A, we conclude that $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$, so $x \in I_n$ for all $n \in \mathbb{N}$.

For uniqueness, note that if x and x' are two real numbers with $x \neq x'$, if we set $\varepsilon := |x - x'| > 0$, there is an interval I_n of length $< \varepsilon$. This interval can thus not contain both x and x'.

6. **Challenge.** In Question 5, we proved that the completeness axiom implies the nested interval principle. Now, prove that the two are actually *equivalent* by showing that the nested interval principle implies the completeness axiom.

Solution. We assume that A is a given non-empty set which is bounded above. We want to show that the supremum $x = \sup(A)$ exists, using only the "nested interval principle". The idea is to construct a decreasing sequence of nested closed intervals $I_n = [a_n, b_n]$ such that

- All b_n are upper bounds for A.
- All a_n are not upper bounds for A.

As explained in the hint to the exercise, we start with $I_1 = [a_1, b_1]$, where a_1 is not an upper bound for A (e.g. $a_1 = y - 1$ for some element $y \in A$) – this exists because A is non-empty – and b_1 is some upper bound for A – this exists because A is bounded above.

Then we define I_n iteratively from I_{n-1} (for $n \geq 2$) as follows: Let m be the midpoint of I_{n-1}, i.e. $m = \frac{a_{n-1}+b_{n-1}}{2}$. We define

$$I_n = [a_n, b_n] := \begin{cases} [a_{n-1}, m] & \text{if } m \text{ is an upper bound for } A, \\ [m, b_{n-1}] & \text{if } m \text{ is not an upper bound for } A. \end{cases}$$

From the nested interval principle, we now know that there exists some x which lies in all intervals $I_n = [a_n, b_n]$, $\forall n \in \mathbb{N}$. [In case you did not know at all how to solve this problem and read until here, try to prove now that x is actually the supremum we are looking for!]

To prove that x is the supremum of A, we need to check the properties of $\sup(A)$:

1. x is an upper bound for A: Assume towards a contradiction that x is not an upper bound for A. Then $\exists y \in A$ with $y > x$ and hence an interval

$I_n = [a_n, b_n]$ of length $b_n - a_n < y - x$. Because $x \in I_n$, we get $b_n - x < y - x$ and thus $b_n < y$. But as $y \in A$ this contradicts the fact that b_n is an upper bound for A.

2. x is the supremum for A: Assume towards a contradiction that there is a smaller upper bound $y < x$. Then there is an interval $I_n = [a_n, b_n]$ of length $b_n - a_n < x - y$. But because $x \in I_n$, we get $x - a_n < x - y$ and thus $a_n > y$. Because y is an upper bound for A, a_n also must be an upper bound for A, which contradicts how we constructed our nested intervals.

The points 1. and 2. together show that x is the supremum $\sup(A)$ which we were looking for.