Lecture 7

6 Regression models with time series errors

In many applications, relationship between two time series Y_t and X_t is of interest. Both series are observed. For example,

- in finance we can try to relate the returns of an individual stock to returns of a market index.
- we can investigate relationship between interest rates of different maturities.

These example lead to a linear regression model of the form

$$Y_t = \alpha + \beta X_t + \varepsilon_t$$

where X_t and Y_t are two time series and ε_t denotes an error term.

OLS method

The parameters of interest α and β can be estimated using the least squares (LS) method. If ε_t is a white noise, then LS estimators are consistent. In practice however, it is common that the errors term ε_t is serially correlated.

In this case we have a regression model with time series errors, and LS estimators of α and β may be not consistent, if the errors are treated as a white noise.

Regression models with time series errors are widely applicable in economics and finance, and often the serial dependence of the errors ε_t is overlooked. Applying this model, we have to study it carefully.

As an example we study the relationship between two US weekly interest rates:

1) r_{1t}, the 1-year Treasury constant maturity rate
2) r_{3t}, the 3-year Treasury constant maturity rate

Both series have 1967 observations from 1962 to 1999, and are measured in percentages.

Figure 2.17 shows the two interest rates:

– solid line denotes 1-year rate
– dashed line denotes 3 -year rate.

Figure 2.18 plots r_{1t} versus r_{3t} indicating strong correlation (linear relationship), which we might expect.

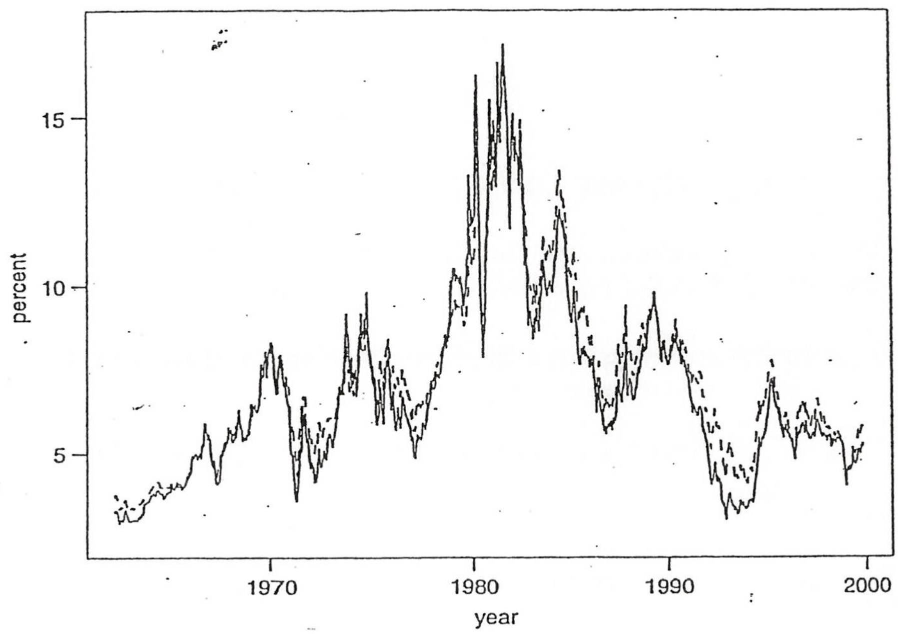

Figure 2.17. Time plots of U.S. weekly interest rates (in percentages) from January 4, 1962 to September 10, 1999. The solid line is the Treasury 1-year constant maturity rate and the dashed line the Treasury 3-year constant maturity rate.

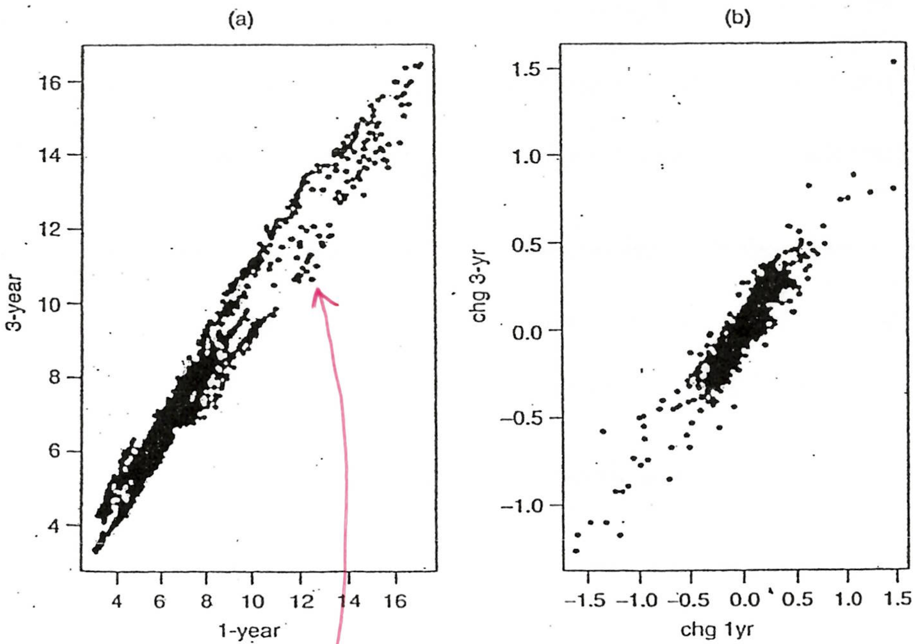

Figure 2.18. Scatterplots of U.S. weekly interest rates from January 5, 1962 to September 10, 1999: (a) 3-year rate versus 1-year rate and (b) changes in 3-year rate versus changes in 1-year rate.

(r_{1t}, r_{3t})

for each t

First naive attempt would be to model the relationship between two rates by the simple model

$$\boxed{r_{3t} = \alpha + \beta r_{1t} + \varepsilon_t.}$$

The fitted model is

$$r_{3t} = 0.911 + 0.924 r_{1t} + \varepsilon_t, \quad \hat{\sigma}_\varepsilon = 0.538.$$

$0.911 > 2(0.032)$
α - significant at 5% level

The standard errors for two coefficients are 0.032 and 0.004. Both coefficients α and β are statistically highly significant. The model confirms the high correlation between the two interest rates.

Fit: Next we have to ask the question: is the model adequate? In case of adequate model the residuals behave a white noise.

Figure 2.19 present the time plot of residuals and the sample ACF of residuals. We see that the ACF of residuals is highly significant and decays slowly: it shows a pattern of non-stationary unit root time series. This indicates that there exists significant difference between these to time series.

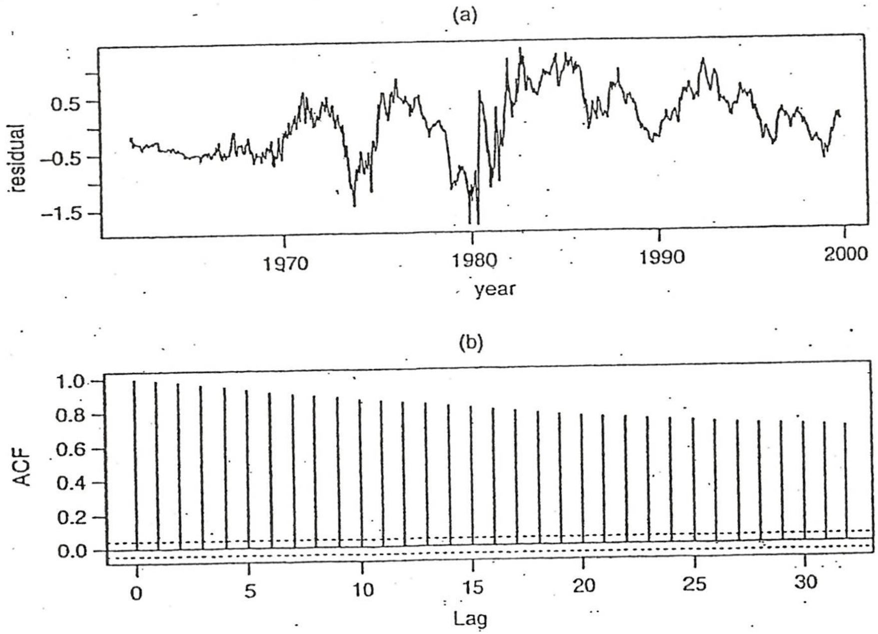

Figure 2.19. Residual series of linear regression (2.44) for two U.S. weekly interest rates: (a) time plot and (b) sample ACF.

Using econometric terminology, if one assumes that these rates are unit root non-stationary, the the behaviour of the residuals indicates that the two interest rates are not cointegrated.

Cointegration would mean that there exist long term equilibrium between two rates, and it would be confirmed if the residuals would behave as a white noise. In our case, the data fail to support the hypothesis of co-integration between two rates. That is not surprising because the interest rates are inversely related to their time to maturities.

The unit root behaviour of interest rates (suggested by the economic theory) and the properties of residuals suggest to consider the series of the interest rates change:

$$c_{1t} = r_{1t} - r_{1,t-1}, \quad \text{changes in 1 -year interest rate}$$

$$c_{3t} = r_{3t} - r_{3,t-1}, \quad \text{changes in 3- year interest rate}$$

Figure 2.20 shows time plots of the change series.

Figure 2.18 (b) provide a scatterplot between these series.

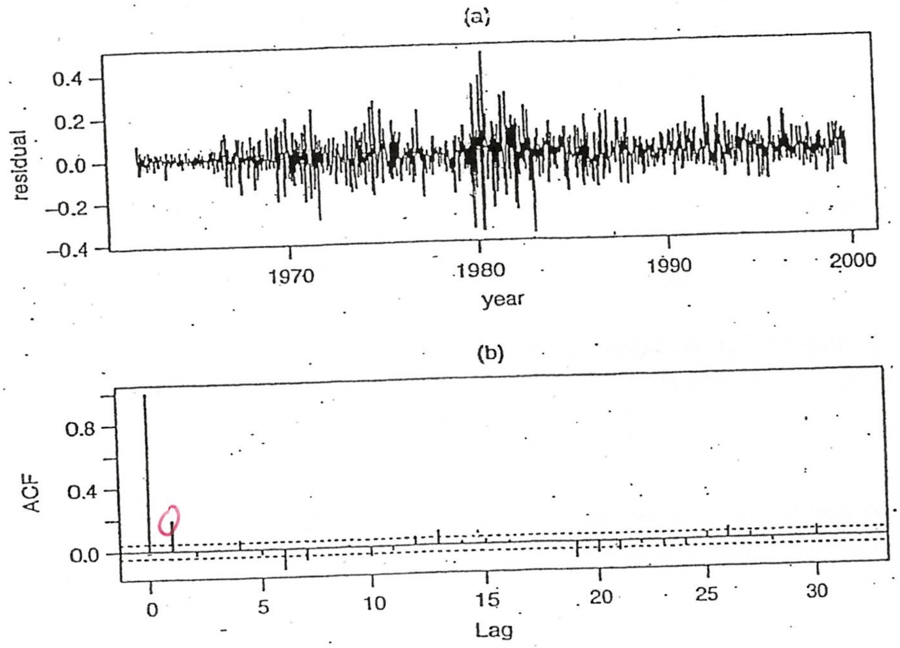

Figure 2.21. Residual series of the linear regression (2.45) for two change series of U.S. weekly interest rates: (a) time plot and (b) sample ACF.

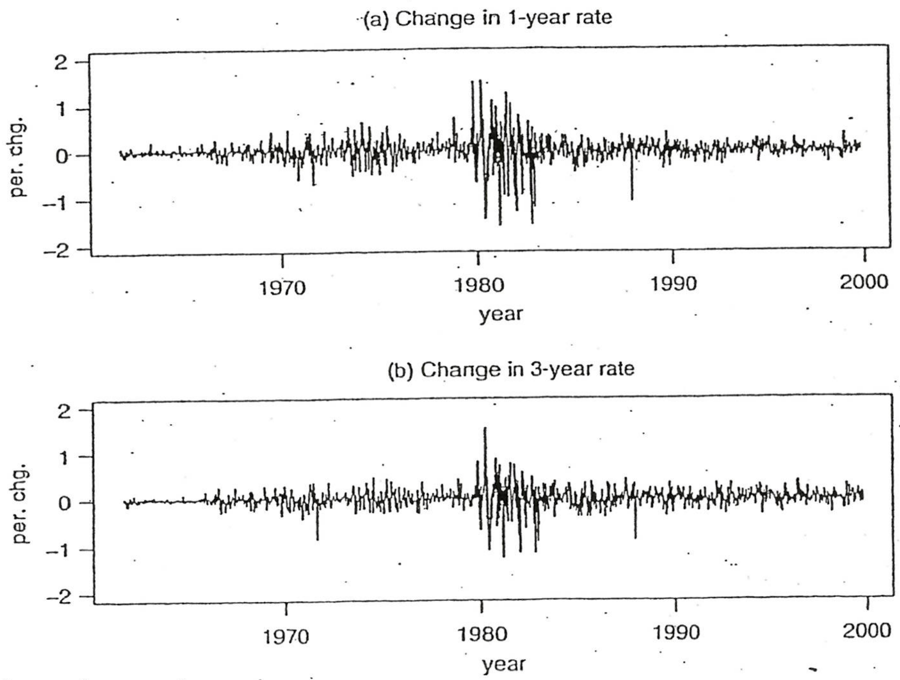

Figure 2.20. Time plots of the change series of U.S. weekly interest rates from January 12, 1962 to September 10, 1999: (a) changes in the Treasury 1-year constant maturity rate and (b) changes in the Treasury 3-year constant maturity rate.

Diagnostic: We fit again a linear regression model:

$$c_{3t} = \alpha + \beta c_{1t} + \varepsilon_t.$$

The fitted model is

$$c_{3t} = 0.0002 + 0.7811 c_{1t} + \varepsilon_t, \qquad \hat{\sigma}_\varepsilon = 0.0682.$$

It shows high correlation between change series. The standard errors of the two coefficients are 0.0015 and 0.0075 which shows that the coefficient

$$\hat{\beta} = 0.7811 > 2\hat{\sigma}_\beta = 2(0.0075) = 0.015$$ significant

is significant at 5 percent significance level, whereas coefficient $\hat{\alpha} = 0.0002$ is not significantly different from zero at this level:

$$\hat{\alpha} = 0.0002 < 2\hat{\sigma}_\alpha = 2(0.0015) = 0.003.$$

Figure 2.21 shows time plot and sample ACF of the residuals. The ACF shows some significant correlation in the residuals, but the correlation is much smaller than before. It indicates that residuals is a weakly dependent time series, which we can model by stationary time series models discussed in previous sections.

Model building. Objective of further discussion: how to build a linear regression model with time series errors?

The approach is very straightforward. We shall estimate the linear regression model and the model of residuals jointly.

FIT MA(1) model

For example, consider the model

$$c_{3t} = 0.002 + 0.7811 c_{1t} + \varepsilon_t, \qquad \hat{\sigma}_\varepsilon = 0.0682.$$

Because residuals are correlated we shall fit a simple ARMA model to residuals. From ACF in Figure 2.21 we specify MA(1) model for residuals and modify the linear regression model to

$$c_{3t} = \alpha + \beta c_{1t} + \varepsilon_t, \qquad \varepsilon_t = e_t - \theta_1 e_{t-1},$$

where e_t is assumed to be a white noise:

Here we use an MA(1) model to capture/model the serial dependence in the error term.

The resulting model is the model of linear regression with time series errors. In practice, more complicated model can be fitted to model the error term.

In practice, if a time series model fitted to the errors is stationary and invertible, than one can estimate the model jointly using maximum likelihood method, using computing packages, see example below. For the US weekly interest rate data, the fitted version is

$$c_{3t} = 0.0002 + 0.7824 c_{1t} + \varepsilon_t, \qquad \varepsilon_t = e_t + 0.2115 e_{t-1}, \qquad \hat{\sigma}_e = 0.0668.$$

The standard error of the parameters are 0.0018, 0.0077 and 0.0221. They show that the constant term α is not significant. This model does not have significant lag-1 ACF residual, but it has some minor correlations at lags 4 and 6.

not significant

Conclusions:

- Comparing models we see, that estimated values of the linear regression parameters might be misleading if the residuals have strong correlation.
- For the change series discussed above, estimated models are close, and adding MA(1) structure to the errors provides only a marginal improvement. There is no surprise in that, since the estimated AR(1) parameter is small numerically: -0.2115, although it is statistically significant (different from zero).
- It is important to check residuals for serial dependence in linear regression analysis.

Because the constant term α is not significant, the model shows that the weekly interest rates are related as

$$\boxed{r_{3t} = r_{3,t-1} + 0.782(r_{1t} - r_{1,t-1}) + e_t + 0.212e_{t-1}.}$$

Summary

A general procedure for analyzing linear regression models with time series is the following:

1. Fit the linear regression model and check for correlation in residuals

2. If residuals are unit-root variables, take the first difference of the variance and check for serial correlation in residuals. If residuals appear to be stationary, identify ARMA models for the residuals, and modify the linear regression model accordingly.

3. Perform a joint maximum likelihood estimation of parameters and check the fitted model for further improvements.

Ljung-Box statistic. Checking for serial correlation of residuals, the Ljung-Box statistic could be used. It tests for correlation at higher order lags.

Durbin-Watson statistic. The Durbin-Watson statistic can be used to test only for the serial correlation at lag-1. For residuals e_t, in case of N observations, the Durbin-Watson statistic is

$$DW = \frac{\sum_{\cancel{j}\,t=2}^{N} (e_t - e_{t-1})^2}{\sum_{\cancel{j}\,t=1}^{N} e_j^2}.$$

DW statistic has asymptotic distribution

$$DW \sim 2(1 - \hat{\rho}_1)$$

where $\hat{\rho}_1$ is the lag -1 ACF of the residuals series e_t.

Conclusion. If there is no serial correlation at lag 1, i.e. $\rho_1 = 0$, then DW takes values close to 2

6.1 Cointegration

Co-integration is generalization of unit roots to vector systems.

Assume that we have two unit root processes y_t and w_t. They satisfies equations

$$y_t - y_{t-1} = \alpha_0 + \alpha_1 \delta_{t-1} + ... + \alpha_p \delta_{t-p}$$

$$w_t - w_{t-1} = \beta_0 + \beta_1 \varepsilon_{t-1} + ... + \beta_q \varepsilon_{t-q}$$

where (δ_t) and (ε_t) are white noise sequences.

In general, a linear combination

$$y_t + a w_t$$

will be also a unit root process (non-stationary process)

Definition. Unit root processes y_t and w_t are said to be co-integrated, if there exists a number a such that a linear combination

$$y_t - \alpha w_t = u_t$$

is a stationary process.

Example 1. Log GDP and log consumption both contain unit roots.

The ratio

$$\frac{\text{GDP}}{\text{consumption}} \sim constant$$

$\log \frac{a}{b} = \log a - \log b$

So

$$\log(GDP) - \log(consumption) = \log(\frac{GDP}{consumption}) \sim \log(constant)$$

and therefore $\log(GDP)$ and $\log(consumption)$ are cointegrated unit root processes.

Example 2. Log stock prices and log dividends both may contain unit roots.

The ratio

$$\frac{\text{price}}{\text{divident}} \sim \textit{stationary process}$$

So

$$\log(\textit{price}) - \log(\textit{divident}) = \log(\textit{stationary}) \sim \textit{stationary}$$

and therefore $\log(\textit{price})$ and $\log(\textit{dividend})$ are co-integrated unit-root time series.

Question. How to estimate cointegration parameter α?

Run regression

$$y_t = \alpha w_t + u_t,$$

use least squares estimator

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} y_i w_i}{\sum_{i=1}^{n} w_i^2}$$

is consistent estimator of α. It converges fast to α at the rate N.