Selected solutions to Problem set 4.

1. Using D'Alembert's formula, we get

$$u(x,t) = \frac{1}{2}\left(e^{x+ct} + e^{x-ct}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin(s)\,ds$$

$$= \frac{e^{x+ct}+e^{x-ct}}{2} - \frac{1}{2c}\cos s \Big|_{x-ct}^{x+ct}$$

$$\cancel{= \frac{e^{x+ct} + e^{x-ct}}{2} \quad \frac{1}{2}}$$

$$= e^x \, \frac{e^{ct}+e^{-ct}}{2} - \frac{1}{2c}\left[\cos(x+ct) - \cos(x-ct)\right]$$

$$= e^x \cosh(ct) + \frac{1}{c}\sin x \cdot \sin(ct)$$

2. $$V_{tt} = U_{xtt} = U_{ttx}$$

$$V_{xx} = U_{xxx}$$

So $$V_{tt} - c^2 V_{xx} = U_{ttx} - c^2 U_{xxx}$$

$$= \frac{\partial}{\partial x}\left(U_{tt} - c^2 U_{xx}\right)$$

$$= \frac{\partial}{\partial x} \cdot 0$$

$$= 0$$

So V also satisfies wave equation

4. ~~$V_{tt} = \frac{\partial}{\partial t}$~~ By Chain Rule.

$$V_t = \partial U_t(\partial x, \partial t)$$

$$V_{tt} = \partial^2 U_{tt}(\partial x, \partial t)$$

$$U_x = \partial U_x(\partial x, \partial t)$$

$$U_{xx} = \partial^2 U_{xx}(\partial x, \partial t)$$

So $V_{tt} - c^2 V_{xx} = \partial^2 (U_{tt} - c^2 U_{xx}) = \partial^2 \cdot 0 = 0$

and V also is a solution to wave equation

7. $$U_t = \frac{c}{2}\left[f'(x+ct) - f'(x-ct)\right] + \frac{1}{2} g(x+ct) + \frac{1}{2} g(x-ct)$$

$$U_{tt} = \frac{c^2}{2}\left[f''(x+ct) + f''(x-ct)\right] + \frac{c}{2} g'(x+ct) - \frac{c}{2} g(x-ct)$$

$$U_x = \frac{1}{2}\left[f'(x+ct) + f'(x-ct)\right] + \frac{1}{2c} g(x+ct) - \frac{1}{2c} g(x-ct)$$

$$U_{xx} = \frac{1}{2}\left[f''(x+ct) + f''(x-ct)\right] + \frac{1}{2c} g'(x+ct) - \frac{1}{2c} g'(x-ct)$$

Combining the above, we get

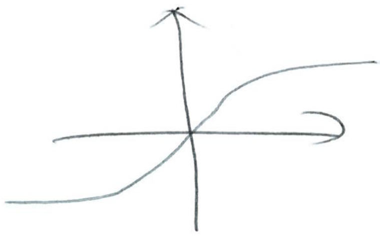
~~Thus~~ $U_{tt} - c^2 U_{xx} = 0$

8. $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{1}{1+s^2}\, ds$

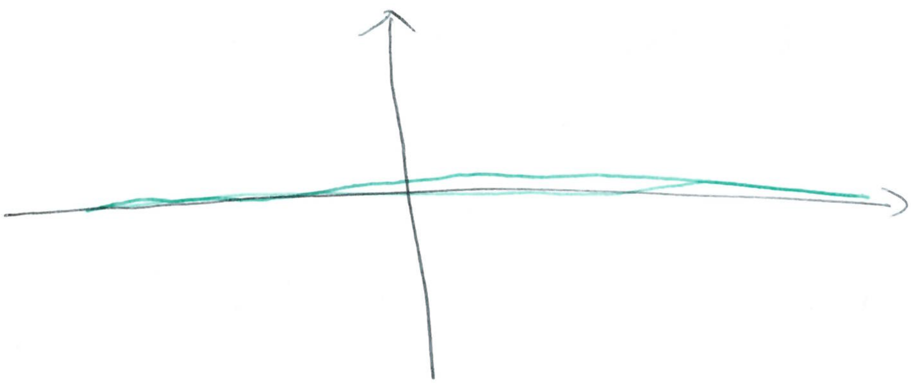
$$= \frac{1}{2c} \arctan x \Big|_{x-ct}^{x+ct}$$

$$= \frac{\arctan(x+ct)}{2c} - \frac{\arctan(x-ct)}{2c}$$

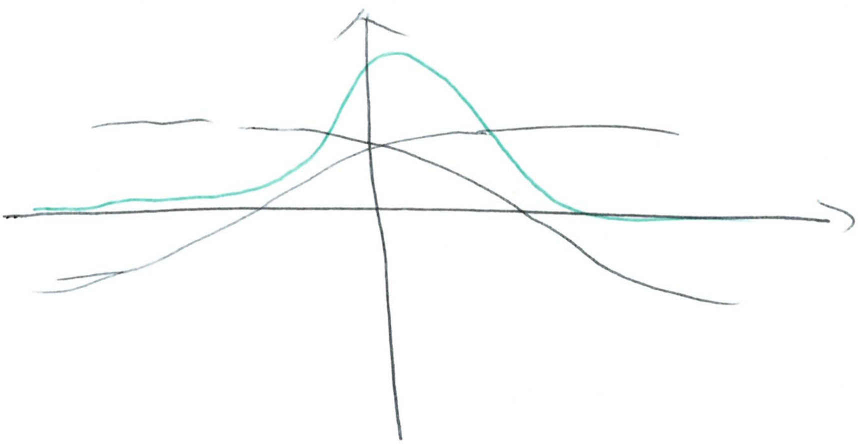
The shape of $\arctan x$ is



At $t=0$



At $t=1$



At $t=5$