Selected Solutions for Problem set 3

2. The charateristic curves are

$$\frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx$$

$$y^2 = x^2 + C.$$

The PDE then becomes (along cheraterstics)

$$y \frac{d}{dx} u(x, y(x)) = 0$$

and is constant along cheraterstic curves.

So $u(x,y) = f(y^2 - x^2)$. $\qquad (*1)$

are general solutions.

When $x=0$, we have $y^2 = C$

$$e^{-C} = e^{-y^2} = u(0, y) = f(y^2) = f(C)$$

So plug in the expression $f(C) = e^{-C}$ into $(*1)$, we get

$$u(x,y) = e^{-(y^2 - x^2)} = e^{x^2 - y^2}$$

3. The chrarateristic curves are given by

$$\frac{dy}{dx} = 1,$$ so they are straight lines

$$y = x + c,$$

and thus $c = y - x$ ~~and y~~

The PDE can then be written so that along chratarstic lines,

$$\frac{d}{dx} u(x, y(x)) + u = e^{x+2y}$$

using e^x as integrating factor, we get

$$\cancel{e^x \frac{d}{dx}}$$

$$e^x \frac{d}{dx} u(x, y(x)) + e^x u = e^{2x+2y}$$

$$\frac{d}{dx} [e^x u(x, y(x))] = e^{2x+2(x+c)}$$

$$\frac{d}{dx} [e^x u(x, y(x))] = e^{4x+2c}.$$

$$e^x u(x, y) = \frac{1}{4} e^{4x+2c} + f(c)$$

~~this is the general solution~~.

~~Now when~~ $$u(x, y) = \frac{1}{4} e^{3x+2c} + e^{-x} f(c)$$ $(*2)$

This is the general solution

Now when $y=0$, we get $c=-x$, and

$$0 = u(x,0) = \frac{1}{4} e^{3x+2c} + e^{-x} \cdot f(c)$$

$$= \frac{1}{4} e^{-3c+2c} + e^{c} f(c).$$

So $f(c) = -\frac{1}{4} e^{-2c}$

~~put~~ plug into (*2) the general solutions,

we get $u(x,y) = \frac{1}{4} e^{3x+2c} + e^{-x} \cdot \left[-\frac{1}{4} e^{-2(y-x)}\right]$

$$= \frac{1}{4} e^{3x+2(y-x)} - \frac{1}{4} e^{-x-2y+2x}$$

$$= \frac{1}{4} e^{x+2y} - \frac{1}{4} e^{x-2y}$$

5. The charateristic curves are solutions to the ODE

$$\frac{dy}{dx} = \frac{y}{1} = y$$

So they are $y(x) = ce^{x}$,

Now given any (x_*, y_*) on the plane $\mathbb{R}^2$, choose c_* to be $c_* = \frac{y_*}{e^{x_*}}$, then the charateristic curve $y = c_* e^{x}$ pass through (x_*, y_*)

7. The charateristics are

$$\frac{dt}{dx} = \frac{2}{1}, \text{ they are straight lines}$$

$$t = 2x + C.$$

Along charateristic lines, PDE becomes

$$\frac{d}{dx}[u(x, t(x))] = u_x + 2u_t = e^{-u}$$

so
$$e^{u} \cdot \frac{d}{dx} u = 1.$$

$$\frac{d}{dx}[e^{u}] = 1$$

$$e^{u} = x + f(c)$$

$$u = \log(x + f(c)) \quad (*3)$$

Now when $t=0$, we have
$c = -2x$ and $x = \frac{c}{-2}$

so

$$1 = u(x,0) = \log(x + f(c)) = \log\left(-\frac{c}{2} + f(c)\right)$$

Exponentiating both sides, get

$$e = -\frac{c}{2} + f(c)$$

$$f(c) = \frac{c}{2} + e$$

plug into $(*3)$, get solution to PDE

$$u(x,t) = \log\left[x + \frac{c}{2} + e\right] = \log\left(x + \frac{t-2x}{2} + e\right)$$

$$= \log\left(\frac{t}{2} + e\right)$$

8. (1) $b^2 = ac$, parabolic !

(2) $b^2 - ac = 1 > 0$, hyperbolic

(3) $b^2 - ac = \frac{1}{4} + 2 > 0$, hyperboic.

(4) $b^2 - ac = 4 - 6 < 0$, elliptic.

9. (2). use $x' = x$,

$y' = -x + y$,

we get $\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$, $\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} + \frac{\partial}{\partial y'}$

$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}$

so PDE becomes

$$2\left(u_{x'x'} + \frac{2\cdot(-6)-4}{2^2}\cdot u_{y'y'}\right) + u_{x'} + u_{y'} = 0$$

& $2u_{x'x'} - 8u_{y'y'} + u_{x'} + u_{y'} = 0$