Finding the MLE of exponential model hazard μ

first define a new variable δ_i where

$\delta_i = 1$ if i^{th} life died

$\delta_i = 0$ otherwise (censored)

Then the likelihood function can be re-written

$$L = \underbrace{\prod_{i=1}^{n}}_{\text{product of probabilities over all } n \text{ observed lives}} \underbrace{f(t_i)^{\delta_i}}_{\text{deaths}} \underbrace{S(t_i)^{1-\delta_i}}_{\text{censored lives}}$$

now in general $f_x(t) = {}_tp_x \mu_{x+t} = S_x(t) \mu_{x+t}$

and in the exponential model here the force of mortality [or hazard] is constant μ

$$\therefore L = \prod_{i=1}^{n} f(t_i)^{\delta_i} S(t_i)^{1-\delta_i}$$

$$= \prod_{i=1}^{n} [S(t_i)\mu]^{\delta_i} S(t_i)^{1-\delta_i}$$

$$= \prod_{i=1}^{n} \mu^{\delta_i} S(t_i)$$

and in the exponential model $S_x(t) = \exp(-\mu t)$

$$\therefore L = \prod_{i=1}^{n} \mu^{\delta_i} \exp(-\mu t_i)$$

We are looking for the MLE $\hat{\mu}$ which maximises L

whichever μ maximises L also maximises $\log L$

$$L = \prod_{i=1}^{n} \mu^{\delta_i} \exp(-\mu t_i)$$

$$\log L = \sum_{i=1}^{n} \delta_i \log \mu - \sum_{i=1}^{n} \mu t_i$$

as $\log a^b = b \cdot \log a$

$\log (dc) = \log d + \log c$

Now differentiate with respect to μ

$$\frac{d}{d\mu} \log L = \frac{1}{\mu} \sum_{i=1}^{n} \delta_i - \sum_{i=1}^{n} t_i$$

setting this to zero to find the maximum gives

$$\frac{1}{\hat{\mu}} \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} t_i$$

$$\hat{\mu} = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} t_i}$$