Coursework 1 Solutions

PS 1: 3(3)

[10 points]

$u_x + e^y u_y = 0$ is a linear homogenous equation

PS 1: 5

[10 points]

$$u_x = f'(x)\, g(y)$$

$$u_y = f(x)\, g'(y)$$

$$u_{xy} = f'(x)\, g'(y)$$

So $$u \cdot u_{xy} = f(x) f(y) \cdot f'(x)\, g'(y)$$
$$= f'(x)\, g(y) \cdot f(x)\, g'(y)$$
$$= u_x \cdot u_y$$

PS 2: 5

[20 points]

By the change of coodinate

$$\begin{cases} \tilde{x} = x + 2y \\ \tilde{y} = 2x - y \end{cases}$$

we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial x} + \frac{\partial}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial x}$$

$$= \frac{\partial}{\partial \tilde{x}} + 2\frac{\partial}{\partial \tilde{y}}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \tilde{x}}\frac{\partial \tilde{x}}{\partial y} + \frac{\partial}{\partial \tilde{y}}\frac{\partial \tilde{y}}{\partial y}$$

$$= 2\frac{\partial}{\partial \tilde{x}} - \frac{\partial}{\partial \tilde{y}}$$

Now the equation becomes

$$\left(\frac{\partial}{\partial \tilde{x}} + 2\frac{\partial}{\partial \tilde{y}}\right)u + 2\left(2\frac{\partial}{\partial \tilde{x}} - \frac{\partial}{\partial \tilde{y}}\right)u + (2x-y)u = (x+2y)(2x-y)$$

$$(1+4)u_{\tilde{x}} + (2-2)u_{\tilde{y}} + \tilde{y}u = \tilde{x}\cdot\tilde{y}$$

$$5u_{\tilde{x}} + \tilde{y}u = \tilde{x}\cdot\tilde{y}$$

Using integrating factor $e^{\frac{\tilde{x}\tilde{y}}{5}}$

we get $e^{\frac{\tilde{x}\tilde{y}}{5}}u_{\tilde{x}} + \frac{\tilde{y}}{5}e^{\frac{\tilde{x}\tilde{y}}{5}}u = \dfrac{e^{\frac{\tilde{x}\tilde{y}}{5}}\cdot\tilde{x}\tilde{y}}{5}$

$$\frac{\partial}{\partial \tilde{x}}\left[e^{\frac{\tilde{x}\tilde{y}}{5}} u\right] = \frac{1}{5} e^{\frac{\tilde{x}\tilde{y}}{5}} \cdot \tilde{x} \cdot \tilde{y}$$

Integrating with respect to $\tilde{x}$ gives

$$e^{\frac{\tilde{x}\tilde{y}}{5}} u = \int \frac{1}{5} e^{\frac{\tilde{x}\tilde{y}}{5}} \tilde{x}\tilde{y}\, d\tilde{x}$$

Using integration by parts, we get

$$e^{\frac{\tilde{x}\tilde{y}}{5}} u = e^{\frac{\tilde{x}\tilde{y}}{5}} \cdot \tilde{x} - \int e^{\frac{\tilde{x}\tilde{y}}{5}} d\tilde{x}$$

$$e^{\tilde{x}\tilde{y}} u = e^{\frac{\tilde{x}\tilde{y}}{5}} \cdot \tilde{x} - \frac{5}{\tilde{y}} e^{\frac{\tilde{x}\tilde{y}}{5}} + f(\tilde{y})$$

$$u(\tilde{x}, \tilde{y}) = \tilde{x} - \frac{5}{\tilde{y}} + f(\tilde{y}) e^{-\frac{\tilde{x}\tilde{y}}{5}}$$

changing the coodinate back, we get

$$u(x, y) = x + 2y - \frac{5}{2x-y} + f(2x-y) \cdot e^{\frac{-2x^2-3xy+2y^2}{5}}$$

PS2: 7 [10 points]

The characteristic curves are

$$\frac{dt}{dx} = -1$$

$$t + x = c$$

so the equation becomes

$$\frac{\partial}{\partial x}u - \frac{\partial t}{\partial x}u - u = 0$$

$$\frac{d}{dx}u = u$$

$$\frac{du}{u} = 1$$

integrating both sides: $\ln u = x + f(c)$

$$u = e^x \cdot \hat{f}(c)$$

$$u(x,t) = e^x \cdot \hat{f}(t+x)$$

Now when $t=0$, we get

from the characteristic equation that

$$x = c$$

So $2 = u(x,0) = e^c \cdot \hat{f}(c)$

we get $\hat{f}(c) = 2 \cdot e^{-c}$

The solution to the initial value problem is thus

$$u(x,t) = e^x \cdot 2 \cdot e^{-c}$$

$$= e^x \cdot 2 \cdot e^{-(x+t)}$$

$$= 2e^{-t}$$

PS 3 : 1

[20 points]

The characteristic curves are

$$\frac{dy}{dx} = \frac{y^2}{x^2}$$

$$\frac{dy}{y^2} = \frac{dx}{x^2}$$

$$\frac{-1}{y} = \frac{-1}{x} + C$$

we get $\quad y = \dfrac{1}{\frac{1}{x} - C}$

and $\quad C = \dfrac{1}{x} - \dfrac{1}{y}$

Along the characteristic curves, one has

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial y}{\partial x} \cdot \frac{\partial u}{\partial y} = u_x + \frac{y^2}{x^2} u_y$$

$$= \frac{x^2 u_x + y^2 u_y}{x^2}$$

$$= \frac{(x+y) u}{x^2}$$

$$= \frac{\left(x + \frac{1}{\frac{1}{x} - c}\right) u}{x^2}$$

$$= \frac{2 - cx}{x(1-cx)} u$$

Divide both sides by u, get

$$\frac{du}{u} = \frac{2-cx}{x(1-cx)}dx$$

$$\frac{du}{u} = \frac{1+ \quad (1-cx)}{x(1-cx)}dx$$

$$\frac{du}{u} = \frac{1}{x(1-cx)}dx + \frac{1}{x}dx$$

$$\frac{du}{u} = \frac{-c}{1-x}dx + \frac{1}{x}dx + \frac{1}{x}dx$$

Integrating both sides, get

$$\ln u = -\ln(1-cx) + 2\cdot\ln x + \hat{f}(c)$$

$$u(x, y(x)) = \frac{x^2}{1-cx}\cdot f(c)$$

Using $\frac{1}{x} - \frac{1}{y} = c$, we get

$$u(x,y) = \frac{x^2}{1-(\frac{1}{x}-\frac{1}{y})\cdot x}\cdot f(\frac{1}{x}-\frac{1}{y})$$

$$= \frac{x^2}{\frac{x}{y}} f(\frac{y-x}{xy})$$

$$= x\cdot y\cdot f(\frac{y-x}{xy})$$

PS 3: 6

[10 points]

The characteristic curves are

$$\frac{dt}{dx} = -1$$

$$x+t = c$$

The equation becomes

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial t}{\partial x}\frac{\partial u}{\partial t} = u_x - u_t = \frac{1}{u^2}$$

so $\quad u^2\,du = dx$

integrating both sides, get

$$\frac{u^3}{3} = x + f(c)$$

$$u^3(x,t) = x + f(x+t)$$

$$u(x,t) = [3x + 3f(x+t)]^{\frac{1}{3}}$$

for any differentiable function f.

PS3: 8(1) [10 points]

$$b^2 - ac = 1^2 - 1^2 = 0$$

The equation is parabolic

PS3: 8(2) [10 points]

$$b^2 - ac = 1^2 - 1 \cdot 0 = 1 > 0$$

The equation is hyperbolic.