Week 3

Theorem $Cov(X,Y) = \mathbb{E}(XY) - \mu_1\mu_2$, where $\mu_1 = \mathbb{E}(X)$, $\mu_2 = \mathbb{E}(Y)$.

Proof. By def-n,

$$Cov(X,Y) = \mathbb{E}((X-\mu_1)(Y-\mu_2))$$
$$= \mathbb{E}(XY - \mu_1 Y - \mu_2 X + \mu_1\mu_2)$$
$$= \mathbb{E}(XY) - \mathbb{E}(\mu_1 Y) - \mathbb{E}(\mu_2 X) + \mathbb{E}(\mu_1\mu_2)$$
$$= \mathbb{E}(XY) - \mu_1 \mathbb{E}(Y) - \mu_2 \mathbb{E}(X) + \mu_1\mu_2$$
$$= \mathbb{E}(XY) - \mu_1\mu_2 - \mu_2\mu_1 + \mu_1\mu_2$$
$$= \mathbb{E}(XY) - \mu_1\mu_2 \qquad \blacksquare$$

Remark. This proof is universal in the sense that it doesn't make any distinction between discrete and continuous r.v.'s.

Particular case. $X = Y$. Then

$$Cov(X,Y) = \mathbb{E}(X-\mu)(X-\mu) = \mathbb{E}(X-\mu)^2 = Var(X),$$ where

$\mu = \mathbb{E}(X) = \mathbb{E}(Y)$. Hence, in this case

$$Cov(X,X) = Var(X) = \mathbb{E}(X^2) - \mu^2.$$

Correlation of X, Y.

Def-n. $\text{Corr}(X,Y) = \dfrac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}}$.

Exercise. Prove that if $a \in \mathbb{R}$, Then:

1) $\text{Cov}(aX, Y) = \text{Cov}(X, aY) = a\,\text{Cov}(X,Y)$

2) $\text{Cov}((X+a), Y) = \text{Cov}(X, (Y+a)) = \text{Cov}(X,Y)$

3) $\text{Cov}((X+Y), Z) = \text{Cov}(X,Z) + \text{Cov}(Y,Z)$

4) $\text{Cov}(X,Y) = \text{Cov}(Y,X)$.

Conditional distribution of continuous random variables.

If X, Y are discrete r.v., then a new r.v. $Z = X \mid Y=y$ can be easily defined. Namely, Z is a r.v. such that

$$\mathbb{P}(Z=x) \overset{def}{=} \mathbb{P}(X=x \mid Y=y) = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)}$$

We want to define a similar r.v. for two jointly cont. r.v.'s X, Y.

But $\mathbb{P}(Y=y) = 0$. nevertheless:

Definition. Given two continuous r.v.'s X, Y with pdf $f_{XY}(x,y)$, we define $Z = X|Y=y$ as a r.v. with pdf

$$f_Z(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \quad -\infty < x < \infty$$

for those y where $f_Y(y) > 0$.

Remarks. 1. $f_Z = f_{X|Y=y}$ 2. $f_{Y|X=x}(y) = \dfrac{f_{XY}(x,y)}{f_X(x)}$.

Example. $f_{X,Y}(x,y) = \begin{cases} e^{-x-y}, & \text{if } x>0, y>0 \\ 0 & \text{otherwise.} \end{cases}$

Find $f_{X|Y=y}$.

Solution. We can find $f_{X|Y=y}$ only for $y > 0$ since $f_Y(y) = 0$ if $y \le 0$.

We found $f_Y(y)$ before.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} e^{-y} & \text{if } y>0 \\ 0 & \text{if } y \le 0. \end{cases}$$

Hence, for $y > 0$,

$$f_{X|Y=y}(x) = \begin{cases} \dfrac{e^{-x-y}}{e^{-y}} = e^{-x} & \text{if } x>0 \\ 0 & \text{if } x \le 0 \end{cases}$$ ◻

<u>Example</u>. $f_{X,Y}(x,y) = \begin{cases} 2e^{-x-y} & \text{if } x \geq y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

Find $f_{X|Y=y}$ — the pdf of $X|Y=y$.

<u>Solution</u>. We find $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

If $y < 0$ then $f_Y(y) = 0$.

If $y \geq 0$ then

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{y}^{\infty} 2e^{-x-y}\,dx$$

$$= 2e^{-y} \int_{y}^{\infty} e^{-x}\,dx = 2e^{-y}\left(-e^{-x}\Big|_{x=y}^{\infty}\right)$$

$$= 2e^{-y}\big(\underbrace{-e^{-\infty}}_{=0} + e^{-y}\big) = 2e^{-2y}$$

<u>Answer</u>. For $y \geq 0$ we have

$$f_{X|Y=y}(x) = \begin{cases} \dfrac{2e^{-x-y}}{2e^{-2y}} = e^{-x+y} & \text{if } x \geq y \\ 0 & \text{otherwise.} \end{cases}$$

<u>Conditional Expectations for Continuous Random Variables</u>

Recall that $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$$E(g(X)) = \int_{-\infty}^{\infty} g(t) f_X(t)\,dt$$

(if $\int_{-\infty}^{\infty} |g(t)| f_X(t)\,dt < \infty$).

If (X, Y) has pdf $f_{XY}(x,y)$, then

$$\mathbb{E}(X \mid Y=y) = \int_{-\infty}^{\infty} t\, f_{X|Y=y}(t)\, dt$$

$$\left(\text{if } \int_{-\infty}^{\infty} |t|\, f_{X|Y=y}(t)\, dt < \infty\right).$$

<u>Example.</u> $f_{X,Y}(x,y) = \begin{cases} 2e^{-x-y} & \text{if } x>y>0 \\ 0 & \text{otherwise} \end{cases}$

Find $\mathbb{E}(X \mid Y=5)$

<u>Solution.</u>

$$f_{X|Y=5}(x) = \begin{cases} e^{-x+5} & \text{if } x>5 \\ 0 & \text{if } x \le 5. \end{cases}$$

Hence

$$\mathbb{E}(X \mid Y=5) = \int_{-\infty}^{\infty} x\, f_{X|Y=5}(x)\, dx$$

$$= \int_5^{\infty} x e^{-x+5}\, dx = e^5 \int_5^{\infty} x\, d(-e^{-x})$$

$$= e^5 \left[-xe^{-x} \Big|_{x=5}^{\infty} + \int_5^{\infty} e^{-x}\, dx \right]$$

$$= e^5 \left(-0 + 5e^{-5} + -e^{-x} \Big|_{x=5}^{\infty} \right)$$

$$= e^5 \left(5e^{-5} + e^{5} \right) = 5 + 1 = 6.$$

Exercise. Compute $\mathrm{Var}(X \mid Y=5)$.

Independence

Independence of events.

Def-n. Two event A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.

More generally, n events $A_1, A_2, \dots A_n$ are independent if

(1) $$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = \mathbb{P}(A_{i_1}) \mathbb{P}(A_{i_2}) \cdots \mathbb{P}(A_{i_k})$$

where $1 \le i_1 < i_2 \dots < i_k \le n$.

(1) is supposed to hold for any subsequence $A_{i_1}, \dots, A_{i_k}$.

Independence of random variables.

Def-n. Two r.v.'s X, Y are independent if for any intervals $[a,b], [c,d]$ the events $\{X \in [a,b]\}$, $\{Y \in [c,d]\}$ are independent.

Remark. X, Y are independent if for any "good" sets $A, B \subset \mathbb{R}$ the event $X \in A$, $Y \in B$ are independent

Remark. Our def-n can be stated as follows. X and Y are independent if

$$P(X \in [a,b] \text{ and } Y \in [c,d]) = P(X \in [a,b]) \times P(Y \in [c,d])$$

for any $[a,b]$, $[c,d]$.

Question. How do we decide whether X, Y are independent?

Theorem. If X, Y are r.v.'s with pdf $f_{X,Y}(x,y)$ then they are independent if and only if

(1) $$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

for all $(x,y) \in \mathbb{R}^2$.

<u>Proof</u>. I. Suppose we know that

$$f_{XY}(x,y) = f_X(x)\, f_Y(y).$$

We then have to show that

$$u \equiv \mathbb{P}(X \in [a,b] \text{ and } Y \in [c,d]) =$$

$$\mathbb{P}(X \in [a,b]) \times \mathbb{P}(Y \in [c,d]).$$

for any $[a,b]$, $[c,d]$. So, by the def-n of pdf

$$u = \int_a^b\int_c^d f_{XY}(x,y)\,dx\,dy = \int_a^b \int_c^d f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_a^b \left(\int_c^d f_X(x) f_Y(y)\,dy \right) dx$$

$$= \int_a^b \Big(f_X(x) \underbrace{\int_c^d f_X(y)\,dy}_{\mathbb{P}(Y \in [c,d])} \Big) dx$$

$$= \mathbb{P}(Y \in [c,d]) \int_a^b f_X(x)\,dx$$

$$= \mathbb{P}(Y \in [c,d]) \times \mathbb{P}(X \in [a,b]). \quad \blacksquare$$

II. We know that X, Y are independent. We have to prove (1). So, consider

$$F_{X,Y}(x,y) = \mathbb{P}(X < x \text{ and } Y < y) =$$

$$\mathbb{P}(X < x)\,\mathbb{P}(Y < y) = F_X(x)\, F_Y(y)$$

Hence we have

$$F_{X,Y}(x,y) = F_X(x)\,F_Y(y)$$

We know that

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\,\partial y}$$

Hence, in our case

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x\,\partial y}\left(F_X(x)F_Y(y)\right)$$

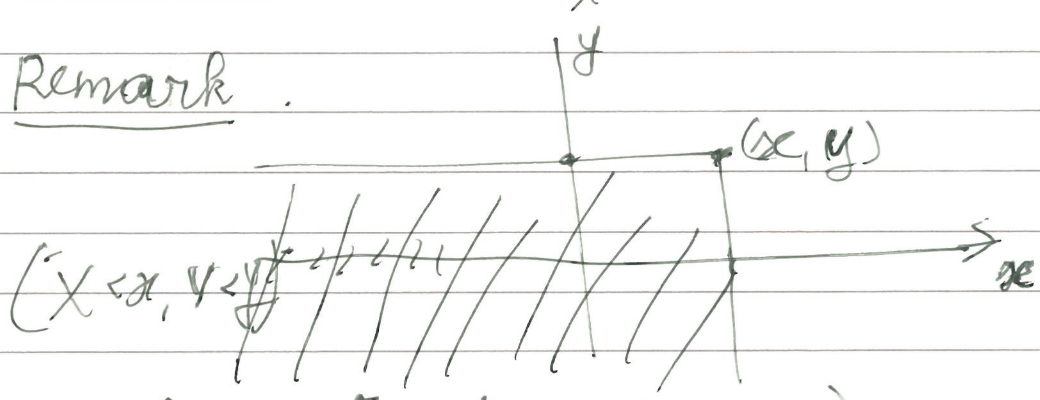
$$= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(F_X(x)F_Y(y)\right)\right]$$

$$= \frac{\partial}{\partial x}\left[F_X(x)\,\underbrace{F_Y'(y)}_{=f_Y(y)}\right]$$

$$= f_Y(y)\,\underbrace{\frac{\partial F_X(x)}{\partial x}}_{f_X(x)} = f_X(x)\,f_Y(y)$$

$\blacksquare$

Remark.



$$(X<x) \equiv (X \in (-\infty, x))$$

$$(Y<y) \equiv (Y \in (-\infty, y))$$

<u>Corollary</u>. If X, Y are independent r.v. with pdf $f_{X,Y}(x,y)$ then

$$f_{X|Y=y}(x) = f_X(x).$$

Indeed, by def-n

$$(2)\quad f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x)\, f_Y(y)}{f_Y(y)} = f_X(x)$$

(for y s.t. $f_Y(y) > 0$).

<u>Remark</u>. We defined $f_{X|Y=y}(x)$ by (2).

Of course, similarly

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = f_Y(y)$$

if they are independent.

<u>Example</u>. $f_{XY}(x,y) = \begin{cases} 6e^{-2x-3y} & \text{if } x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$

Are X, Y independent?

<u>Solution</u>. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \begin{cases} 2e^{-2x}, & x>0 \\ 0 & \text{otherwise} \end{cases}$

Similarly

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \begin{cases} 3e^{-3y} & , \text{if } y>0 \\ 0 & , \text{if } y \le 0. \end{cases}$$

So we see that

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

Indeed, if $x>0$, $y>0$ then

$$f_X(x) f_Y(y) = 6e^{-2x-2y} = f_{X,Y}(x,y)$$

If $x \le 0$ or $y \le 0$ then

$$f_X(x) f_Y(y) = 0 = f_{X,Y}(x,y)$$

$\boxtimes$

<u>Theorem</u>. Let X, Y be r.v.'s with pdf $f_{XY}(x,y)$. Then X, Y are independent if and only if there are two functions g and h such that

$$f_{XY}(x,y) = g(x) h(y)$$

for all x, y.

<u>Proof</u>. I. If X, Y are independent then

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

(by the previous theorem). So we can set $g(x) = f_X(x)$, $h(y) = f_Y(y)$. $\square$

Tutorial Week 3

$f_{X|Y=y}(x) \qquad Z = X|Y=y$

$$\mathbb{E}(g(Z)) = \int_{-\infty}^{\infty} g(x) f_{X|Y=y}(x)\,dx$$

$$\mathbb{E}(Z^2) = \int_{-\infty}^{\infty} x^2 f_{X|Y=5}(x)\,dx$$

We found $f_{X|Y=5}(x) = \begin{cases} e^{-x+5} & \text{if } x>5 \\ 0 & \text{if } x \le 5 \end{cases}$

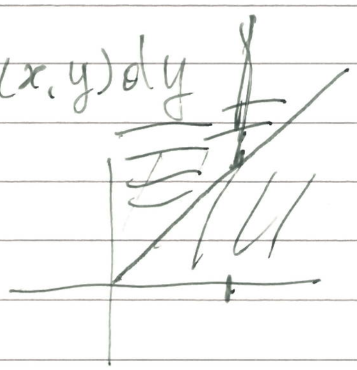
$$\mathrm{Var}(X|Y=y) = \mathbb{E}(Z^2) - (\mathbb{E}(Z))^2 =$$

$$\mathbb{E}\left((Z-\mathbb{E}(Z))^2\right)$$

In our case

$$\mathbb{E}(Z^2) = \int_{-\infty}^{\infty} x^2 f_{X|Y=5}(x) = \int_5^{\infty} x^2 e^{-x+5}\,dx$$

Problem 8(a) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_0^{\infty} dx \int_0^{\infty} f_{X,Y}(x,y)\,dy$



$$= \int_0^{\infty} dx \int_x^{\infty} c e^{-2x-3y}\,dy$$

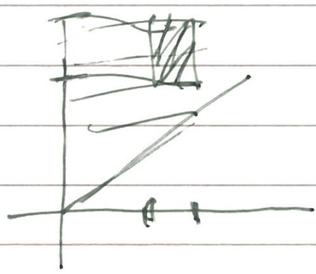
$$c\int_x^{\infty} e^{-2x} e^{-3y}\,dy = ce^{-2x} \times \left(-\frac{1}{3} e^{-3y}\Big|_{y=x}^{\infty}\right)$$

$$= ce^{-2x}\left(-\frac{1}{3}e^{-3\times\infty} + \frac{1}{3}e^{-3x}\right) = \frac{1}{3}ce^{-5x}.$$

Next,

$$\frac{c}{3}\int_0^\infty e^{-5x}\,dx = \frac{c}{15}\left(-e^{-5x}\Big|_{x=0}^{\infty}\right) = \frac{c}{15} = 1.$$

$$\Rightarrow \quad c = 15.$$



Problem 8 (b). (i)

$P(2 < X < 3$ and $7 < Y < 9)$

$$= \int_2^3 dx \int_7^9 dy\, 15e^{-2x-3y} = \dots,$$

(ii) $P(5 < X < 6$ and $5 < Y < 6)$

$$= \int_5^6 dx\left(\int_5^6 f_{XY}(x,y)\,dy\right)$$

$$= \int_5^6 dx \int_x^6 15e^{-2x-3y}\,dy$$

$$= 15\int_5^6 e^{-2x}\left(-\frac{1}{3}e^{-3y}\Big|_{y=x}^{6}\right)$$

$$= \frac{15}{3}\int_5^6 e^{-2x}\left(e^{-3x}-e^{-18}\right)dx$$

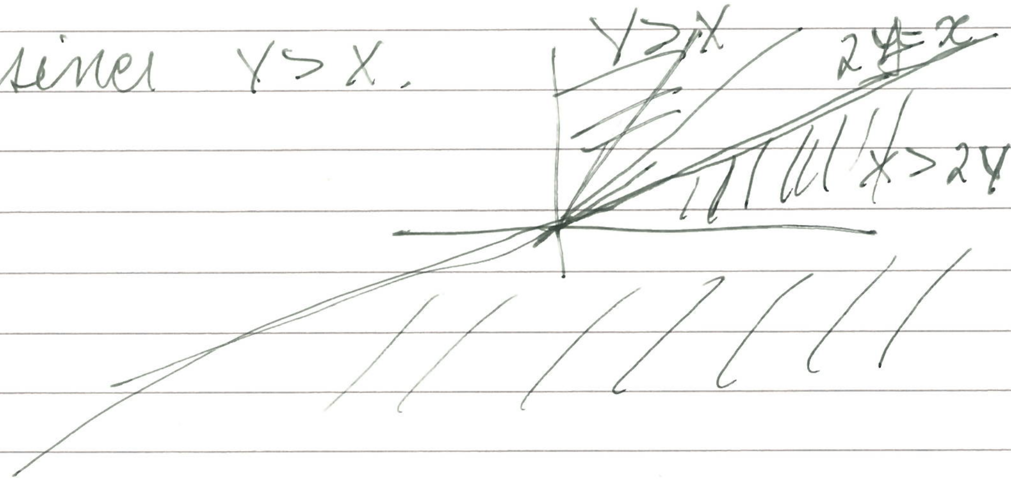
$$= 5\left(-\frac{1}{5}e^{-5x}\Big|_{x=5}^{6} - e^{-18}\cdot\left(\frac{-1}{2}\right)e^{-2x}\Big|_{x=5}^{6}\right)$$

$$5\left(\frac{1}{5}e^{-25} - \frac{1}{5}e^{-30} + \frac{1}{2}e^{-18}e^{-12} - \frac{1}{2}e^{-18}\times e^{-10}\right)$$

(3)

(iii). $\mathbb{P}(X > 2Y > 0) = 0$,

since $Y > X$.



$\mathbb{P}(Y > 2X > 0) = ?$