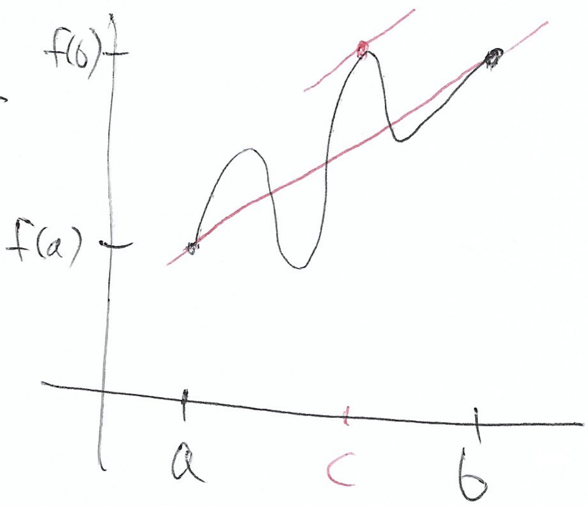
A useful tool in proving the Theorem is :

Mean Value Theorem

If $f:[a,b] \to \mathbb{R}$ is C^1 (on (a,b))
there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



Proof of Theorem We shall prove the attracting case, leaving the repelling case as an exercise.

Suppose $|f'(p)| < 1$.
We must show that p is attracting, i.e. that there exists $\delta > 0$ such that if $x \in (p-\delta, p+\delta) =: I$ such that if $x \in I$ then $\lim_{n\to\infty} f^n(x) = p$.

Since $|f'(p)| < 1$, let us choose some $K \in (|f'(p)|, 1)$.

But since f' is continuous (i.e. f is C^1) we can choose $\delta > 0$ such that if $I = (p-\delta, p+\delta)$ then

$$|f'(x)| < K \quad \text{for all } x \in I \quad (*)$$

Now by the Mean Value Theorem, for all $x \in \mathbb{R} \setminus \{p\}$

$$f(x) - f(p) = f'(c)(x-p) \quad (**)$$

for some $c \in \begin{cases} (p, x) & \text{if } x > p \\ (x, p) & \text{if } p > x \end{cases}$

Therefore for all $x_0 \in I \setminus \{p\}$,

$$|f(x_0) - f(p)| \overset{\text{by } (**)}{=} |f'(c)|\,|x_0 - p|$$

$$\overset{\text{by } (*)}{<} K\,|x_0 - p|$$

i.e. $|x_1 - p| < K\,|x_0 - p|$ $(***)$

In particular,

$$|x_1 - p| \overset{\text{since } K<1}{<} |x_0 - p| \overset{\text{since } x_0 \in I = (p-\delta, p+\delta)}{<} \delta$$

therefore $x_1 \in I = (p-\delta, p+\delta)$

Applying the Mean Value Theorem again (on $[p, x_1]$ or $[x_1, p]$) gives:

$$|f(x_1) - f(p)| < K\,|x_1 - p|$$

i.e. $|x_2 - p| < K\,|x_1 - p|$

$$\overset{\text{by } (***)}{<} K^2\,|x_0 - p|$$

Repeating this argument inductively we deduce

$$|x_n - p| < K^n\,|x_0 - p| \quad \forall n \in \mathbb{N}$$

Taking limits as $n \to \infty$, and recalling that $K \in (0,1)$ (so that $\lim_{n\to\infty} K^n = 0$) we get that

$$\lim_{n\to\infty} |x_n - p| = 0$$

i.e. $\lim_{n\to\infty} x_n = p$

i.e. $\lim_{n\to\infty} f^n(x_0) = p$

So p is attracting. $\square$

We can generalise the notions of 'attracting' and 'repelling' to periodic points:

<u>Defn</u> Suppose $p \in \mathbb{R}$ is a periodic point of least period k for the function $f : \mathbb{R} \to \mathbb{R}$ (so $f^k(p) = p$).

We say p is <u>attracting</u> (for f) if p is an attracting fixed point for f^k.

We say p is <u>repelling</u> (for f) if p is a repelling fixed point for f^k.

It can be shown that if a point p (of least period k under f) is attracting then so are the other points in its orbit, i.e. so are $f(p), f^2(p), \ldots, f^{k-1}(p)$.
So we call this an <u>attracting periodic orbit</u>, or an <u>attracting cycle</u>.
Similarly, if p is repelling then so are $f(p), \ldots, f^{k-1}(p)$, so we call $\{p, f(p), \ldots, f^{k-1}(p)\}$ is a <u>repelling periodic orbit</u> or <u>repelling cycle</u>.

A consequence of the previous theorem is:

<u>Corollary</u> If $f : \mathbb{R} \to \mathbb{R}$ is C^1, and p is a periodic point of least period k, then

- p is attracting if $|(f^k)'(p)| < 1$
  (i.e. $\{p, f(p), \dots, f^{k-1}(p)\}$ is attracting if $|(f^k)'(p)| < 1$)
- and p is repelling if $|(f^k)'(p)| > 1$
  (i.e. $\{p, f(p), \dots, f^{k-1}(p)\}$ is a repelling periodic orbit if $|(f^k)'(p)| > 1$)

Definition If x_0 is a periodic point of least period k (i.e. $x_0, x_1, \ldots, x_{k-1}$ is a k-cycle) then we refer to the value $(f^k)'(x_0)$ as its multiplier.

We can show that

$$(f^k)'(x_0) = f'(x_0) f'(x_1) \ldots f'(x_{k-1})$$
$$= \prod_{i=0}^{k-1} f'(x_i) \qquad (*)$$

i.e. The multiplier is equal to the product of the derivatives (of f) at all points in the orbit.
So the multiplier is the same value for all points in the orbit.

To justify (*), first note that

$$(f^2)'(x) = (f\circ f)'(x)$$

chain rule

$$= f'(f(x)).f'(x)$$

so $$(f^2)'(x_0) = f'(f(x_0)).f'(x_0)$$
$$= f'(x_1)f'(x_0)$$

(which is (*) for $k=2$).

Similarly for $k=3$:

$$(f^3)'(x) = (f\circ f\circ f)'(x)$$

chain rule

$$= f'(f\circ f(x)).(f\circ f)'(x)$$

by the above

$$= f'(f^2(x)).f'(f(x)).f'(x)$$

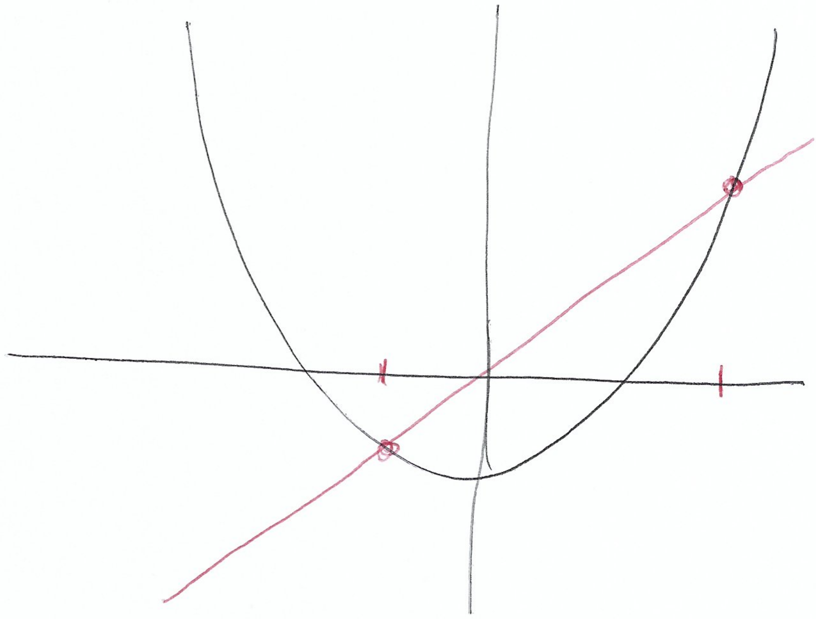
so $$(f^3)'(x_0) = f'(f^2(x_0))\, f'(f(x_0))\, f'(x_0)$$
$$= f'(x_2)\, f'(x_1)\, f'(x_0)$$

To prove (*) in general we can use induction (Exercise).

Example $f(x) = x^2 - 1$

Recall the fixed points are at

$$x = \frac{1 \pm \sqrt{5}}{2}$$



Note that $f'(x) = 2x$

So $f'\left(\frac{1+\sqrt{5}}{2}\right) = 1+\sqrt{5} > 1$,

thus the fixed point $\frac{1+\sqrt{5}}{2}$ is repelling (by the Theorem).

For the other fixed point $\frac{1-\sqrt{5}}{2}$, note that $f'\left(\frac{1-\sqrt{5}}{2}\right) = 1-\sqrt{5}$, which is larger than 1 in absolute value, so this fixed point is repelling as well, again by the Theorem.

We also saw that $\{-1, 0\}$ is a 2-cycle.

We can use the Corollary to determine whether this 2-cycle is attracting or repelling.

$$f'(-1) = 2(-1) = -2$$

$$f'(0) = 2.0 = 0$$

So the multiplier for this 2-cycle is

$$(f^2)'(0) = f'(0)\,f'(-1)$$

$$= 0.(-2) = 0,$$

which < 1, therefore this 2-cycle is attracting.

therefore $|(f^2)'(0)| = |0| < 1$

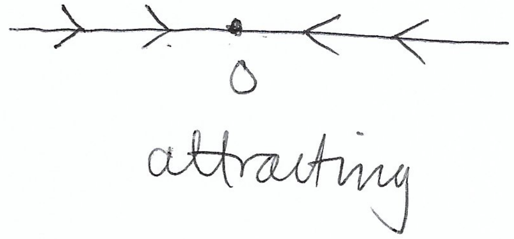
__Remark__ If a fixed point, or a k-cycle, has multiplier equal to $+1$ or -1, then we cannot immediately tell whether it is attracting, or repelling, or neither.

We __could__ examine higher order derivatives to determine the behaviour near this k-cycle

__Examples__ For each of the following ~~examples~~ functions f, the point 0 is a fixed point and $f'(0) = 1$

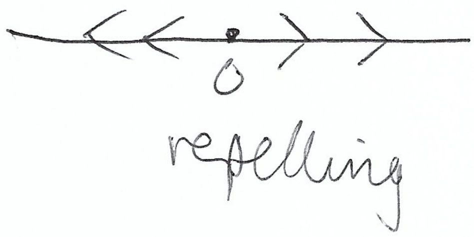
(1) $f(x) = x - x^5$

Here 0 is attracting



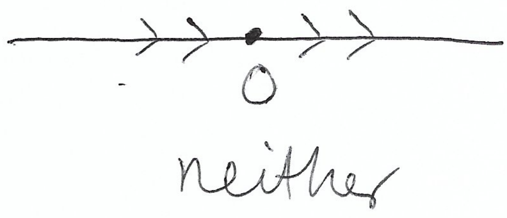
(2) $f(x) = x + x^3$

Here 0 is repelling



(3) $f(x) = x + x^2$

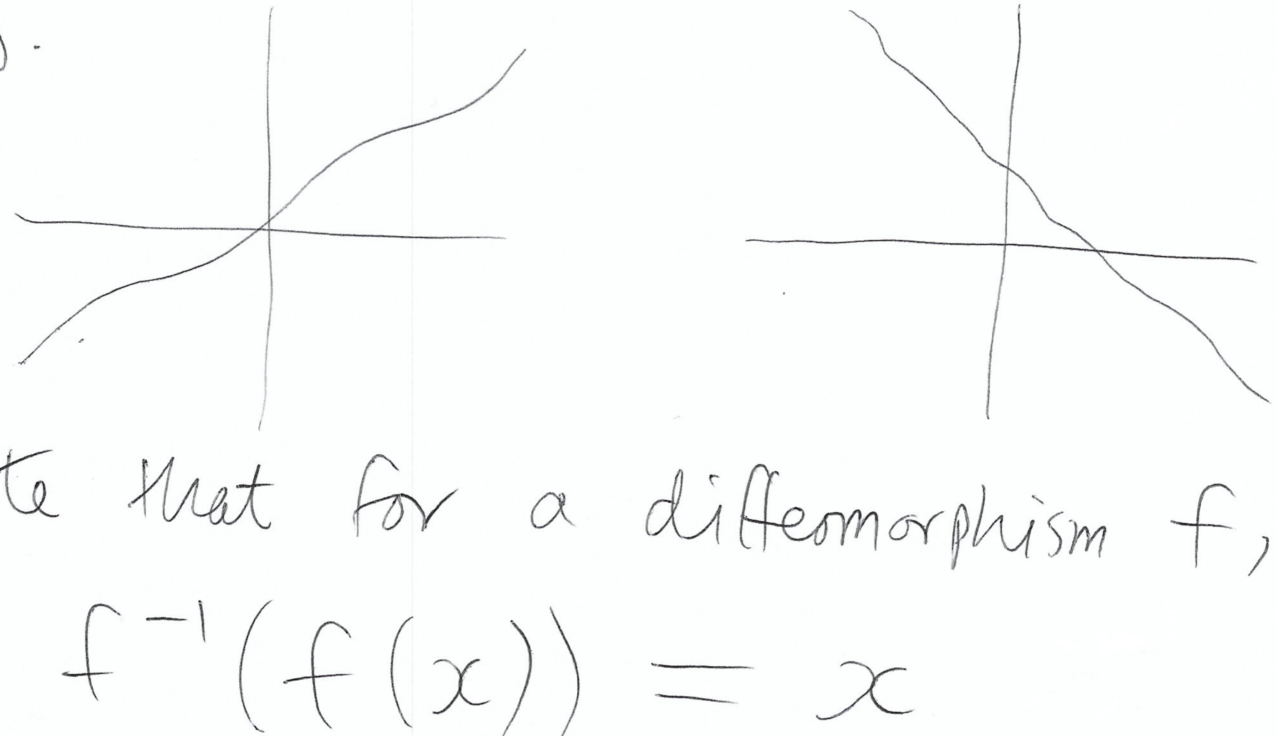
Here 0 is neither attracting nor repelling



Diffeomorphisms

<u>Defn</u> A <u>diffeomorphism</u> (or C^1-diffeomorphism) is a bijective (i.e. invertible) function $f : \mathbb{R} \to \mathbb{R}$ for which both f and f^{-1} are C^1 (i.e. both f and f^{-1} are differentiable with continuous derivative)

e.g.



Note that for a diffeomorphism f,

$$f^{-1}(f(x)) = x$$

Differentiating both sides we get:

$$(f^{-1})'(f(x)) \cdot f'(x) = 1$$

So $$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

We see that neither f' nor $(f^{-1})'$ can be equal to zero at any point.

~~Since~~ So by the Intermediate Value Theorem applied to the continuous function f', we see that:

either $f'(x) > 0$ for all $x \in \mathbb{R}$

or $f'(x) < 0$ for all $x \in \mathbb{R}$

<u>Defn</u> We say that $f : \mathbb{R} \to \mathbb{R}$ is <u>order-preserving</u> if whenever $a < b$ then $f(a) < f(b)$, and we say it is <u>order-reversing</u> if whenever $a < b$ then $f(a) > f(b)$.

__Lemma__ If $f:\mathbb{R}\to\mathbb{R}$ is a diffeomorphism then it is either order-preserving or order-reversing.
If $f'>0$ then f is order-preserving, and if $f'<0$ then f is order-reversing.

__Proof__ ~~let~~ Suppose $a,b\in\mathbb{R}$, with $a<b$.
If $f'(x)>0$ for all $x\in\mathbb{R}$,
then

$$\frac{f(b)-f(a)}{b-a}=f'(c)$$

for some $c\in(a,b)$ by ~~M~~ the Mean Value Theorem.

So

$$\frac{f(b)-f(a)}{b-a}>0$$

i.e. $f(b)-f(a)>0$

i.e. $f(b)>f(a)$

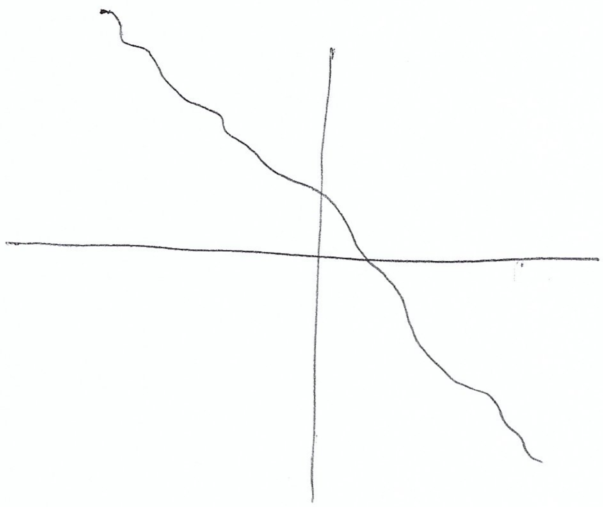
So f is order-preserving.

A similar argument shows that if $f'<0$ then f is order-reversing. $\square$

Proposition If $f: \mathbb{R} \to \mathbb{R}$ is an order-reversing diffeomorphism then f has exactly one fixed point.

Proof First we show that f has a fixed pt.
Since f is order-reversing, we know that if $a < b$ then $f(a) > f(b)$.
Now $f: \mathbb{R} \to \mathbb{R}$ is a bijection, so

$$\lim_{x \to \infty} f(x) = -\infty$$



and

$$\lim_{x \to -\infty} f(x) = +\infty$$

Now consider the function $g(x) := f(x) - x$.
Note that $\lim_{x \to \infty} g(x) = -\infty$

and $\lim_{x \to -\infty} g(x) = +\infty$

By the Intermediate Value Theorem there exists $p \in \mathbb{R}$ such that $g(p) = 0$, i.e. $f(p) = p$, i.e. p is a fixed pt for f.

We next show that p is the <u>unique</u> fixed point.

Suppose (in order to obtain a contradiction) that there is some other fixed point, i.e. some value q, with $q \neq p$, such that $f(q) = q$.

Without loss of generality suppose $q < p$. $(**)$

Since f is order-reversing we have that $f(q) > f(p)$. $(*)$

But $f(q) = q$ and $f(p) = p$, so $(*)$ says that $q > p$, which contradicts $(**)$.

This is the required contradiction, so indeed there is a unique fixed point. $\square$

Note that for order-preserving diffeomorphisms we cannot make any general statements about the number of fixed points

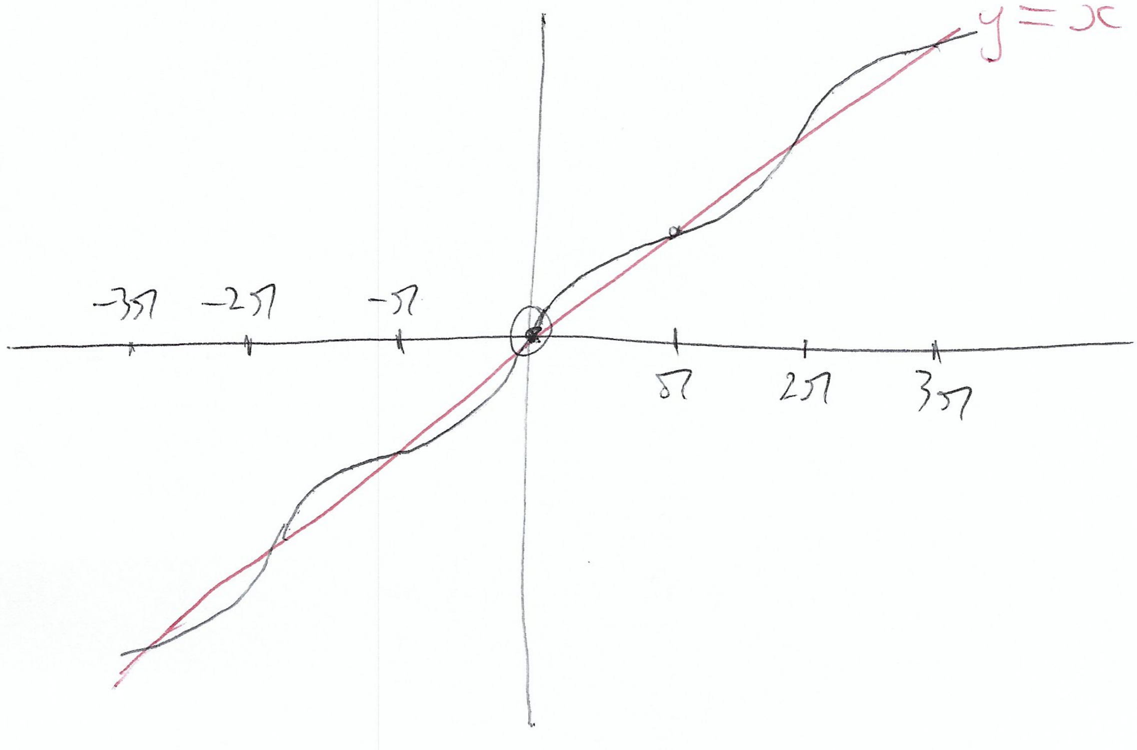
Example $f(x) = x$

Here every real number is a fixed point.

Example $f(x) = x + c$, where $c \neq 0$

Here there are no fixed points.

Example $f(x) = x + \frac{1}{2}\sin(x)$



The fixed point equation is $f(x) = x$

i.e. $x + \frac{1}{2}\sin(x) = x$

i.e. $\frac{1}{2}\sin(x) = 0$

So $x = n\pi$, $n \in \mathbb{Z}$, are fixed points.

Note that $f'(x) = 1 + \frac{1}{2}\cos(x)$

Thus $f'(x) \geq \frac{1}{2} > 0$ for all $x \in \mathbb{R}$

So f is an order-preserving diffeomorphism.

We can check if the fixed points at $x = n\pi$, $n \in \mathbb{Z}$, are attracting or repelling.

$$f'(n\pi) = 1 + \tfrac{1}{2}\cos(n\pi)$$

$$= \begin{cases} 3/2 & \text{if } n \text{ is even} \\ 1/2 & \text{if } n \text{ is odd} \end{cases}$$

So (by the theorem proved yesterday), the fixed point $n\pi$ is repelling if n is even, and attracting if n is odd.