Exercise sheet 3 solutions

1. For the first part of the question, we saw in the lectures that with a Gamma(α, β) prior distribution for λ, the posterior distribution for λ is Gamma$(n+\alpha, S+\beta)$, where $S = \sum_{i=1}^n t_i$.

   The observed data for this part have $n = 6$ and $S = 55$. So with $\alpha = 0.1$ and $\beta = 2$, the posterior distribution is Gamma$(6.1, 57)$.

   For the second part of the question, we can reorder the data so that the data-point where the exact time was not observed is the last one. There are $m = 5$ data-points with the exact time known. The likelihood function is

   $$p(t \mid \lambda) = \lambda^m e^{-\lambda S}.$$

   A gamma distribution as a prior for λ with parameters α and β has probability density function

   $$p(\lambda) = \frac{\beta^\alpha \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)}, \ \lambda > 0$$

   So the posterior distribution is

   $$\begin{aligned} p(\lambda \mid t) &\propto p(\lambda)\, p(t \mid \lambda) \\ &\propto \lambda^{\alpha-1} e^{-\beta\lambda}\, \lambda^m e^{-\lambda S} \\ &= \lambda^{m+\alpha-1} e^{-(S+\beta)\lambda} \end{aligned}$$

   So the posterior distribution is now Gamma$(m + \alpha, S + \beta)$. The data has changed to give $S = 56$, so the posterior distribution is Gamma$(5.1, 58)$.

2. The datapoints follow a Poisson distribution, which is discrete, so the likelihood contribution for each datapoint is the Poisson probability mass function

   $$p(y_i \mid \lambda) = \frac{\lambda^{y_i} e^{-\lambda}}{y_i!}$$

   5 The overall likelihood is

   $$p(y \mid \lambda) = \prod_{i=1}^n \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} = \frac{\lambda^S e^{-n\lambda}}{\prod_{i=1}^n y_i!}, \text{ where } S = \sum_{i=1}^n y_i.$$

   The log-likelihood is

   $$\ell(\lambda; y) = S \log(\lambda) - n\lambda - \log\left(\prod_{i=1}^n y_i!\right).$$

   Differentiating and setting to zero gives

   $$\frac{d\ell}{d\lambda} = \frac{S}{\lambda} - n = 0$$

   Hence the MLE is

   $$\hat{\lambda} = \frac{S}{n} = \bar{y}.$$

For the data in the question, $n = 6$ and $S = 24$, so the MLE is

$$\hat{\lambda} = \frac{24}{6} = 4.$$

A gamma distribution as a prior for λ with parameters α and β has probability density function

$$p(\lambda) = \frac{\beta^\alpha \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)}, \ \lambda > 0$$

Multiplying the prior and likelihood, the posterior distribution has pdf

$$\begin{aligned} p(\lambda \mid y) &\propto p(\lambda)\, p(y \mid \lambda) \\ &= \frac{\lambda^S e^{-n\lambda}}{\prod_{i=1}^n y_i!} \frac{\beta^\alpha \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)} \\ &\propto \lambda^S e^{-n\lambda} \lambda^{\alpha-1} e^{-\beta\lambda} = \lambda^{S+\alpha} e^{-(n+\beta)\lambda}. \end{aligned}$$

$p(\lambda \mid y)$ is proportional to a $\text{Gamma}(S+\alpha, n+\beta)$ pdf, hence the posterior distribution for λ must be $\text{Gamma}(S+\alpha, n+\beta)$.

The posterior distribution is in the same family of distributions as the prior, hence the family of gamma distributions is conjugate to the Poisson likelihood.

With $\alpha = 1, \beta = 1$, the posterior distribution is $\text{Gamma}(25, 7)$.

3, 4. There is R code for these questions on QMPlus.