Queen Mary University of London

MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 1 (Demon Games)

Consider the following five statements:

(a) $\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}\ :\ n = m^2$.

(b) $\forall a \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}, |x - a| < \delta\ :\ |3x - 3a| < \varepsilon$.

(c) $\forall x \in \mathbb{Q}\ \forall y \in \mathbb{Q}, y \neq x\ \exists z \in \mathbb{Q}\ :\ (x < z < y)$ or $(y < z < x)$.

(d) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}, |x| < \delta\ :\ x^2 < \varepsilon$.

(e) $\forall N \in \mathbb{N}\ \exists n \in \mathbb{N}, n \geq N\ \exists m \in \mathbb{N} : n = m!$.

Questions.

1. Write down the Demon games corresponding to (a)-(e)

   Solution. Recall that the quantifier $\forall$ corresponds to the Demon picking, and $\exists$ to us picking. If the final statement is true then we win (and otherwise the Demon wins). So the Demon Games are the following.

   (a) First the Demon picks $n \in \mathbb{N}$.
   Then we pick $m \in \mathbb{N}$.
   We win if $n = m^2$ (otherwise the Demon wins).

   (b) First the Demon picks $a \in \mathbb{R}$.
   Then the Demon picks $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$.
   Then we pick $\delta \in \mathbb{R}$ with $\delta > 0$.
   Then the Demon picks $x \in \mathbb{R}$ with $|x - a| < \delta$.
   We win if $|3x - 3a| < \varepsilon$.

   (c) The Demon picks $x \in \mathbb{Q}$
   The Demon picks $y \in \mathbb{Q}$ with $y \neq x$.
   Then we pick $z \in \mathbb{Q}$.
   We win if $x < z < y$ or $y < z < x$ (i.e., if z lies between x and y).

   (d) The Demon picks $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$.
   Then we pick $\delta \in \mathbb{R}$ with $\delta > 0$.
   Then the Demon picks $x \in \mathbb{R}$ with $|x| < \delta$.
   We win if $x^2 < \varepsilon$.

(e) The Demon picks $N \in \mathbb{N}$.
We pick $n \in \mathbb{N}$ with $n \geq N$.
We pick $m \in \mathbb{N}$.
We win if $n = m!$.

2. Write down the *negation* of each of the statements (a)-(e).

Solution. The negations are:

(a) $\exists n \in \mathbb{N}\ \forall m \in \mathbb{N}\ :\ n \neq m^2$.
(b) $\exists a \in \mathbb{R}\ \exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in \mathbb{R}, |x - a| < \delta\ :\ |3x - 3a| \geq \varepsilon$.
(c) $\exists x \in \mathbb{Q}\ \exists y \in \mathbb{Q}, y \neq x\ \forall z \in \mathbb{Q}\ :\ (z \leq x \vee z \geq y) \wedge (z \leq y \vee z \geq x)$. (By Boolean algebra, the formula $(z \leq x \vee z \geq y) \wedge (z \leq y \vee z \geq x)$ is equivalent to $(z \leq x \wedge z \leq y) \vee (z \geq x \wedge z \geq y)$, which is easier to understand.)
(d) $\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in \mathbb{R}, |x| < \delta\ :\ x^2 \geq \varepsilon$.
(e) $\exists N \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq N\ \forall m \in \mathbb{N} : n \neq m!$.

Explanation. Recall that negating a statement corresponds to swapping the role with the Demon in the corresponding Demon Game. Let us look at this once more, for example for the statement (d). The statement was

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}, |x| < \delta\ :\ x^2 < \varepsilon.$$

which corresponds to the Demon Game:

The Demon picks $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$.
Then we pick $\delta \in \mathbb{R}$ with $\delta > 0$.
Then the Demon picks $x \in \mathbb{R}$ with $|x| < \delta$.
We win if $x^2 < \varepsilon$.

Now we change the role with the Demon, so the new game is now the following.

First we pick $\varepsilon > 0$.
Then the Demon picks $\delta > 0$.
Then we pick $x \in \mathbb{R}$ with $|x| < \delta$.
The Demon wins if $x^2 < \varepsilon$ (and hence we win if $x^2 \geq \varepsilon$).

This can then be translated back to the mathematical statement

$$\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in \mathbb{R}, |x| < \delta\ :\ x^2 \geq \varepsilon.$$

This means that the negation of a mathematical expression can be obtained by changing all the quantifiers $\exists$ into $\forall$ and all $\forall$ into $\exists$ and negating the final statement. Be careful that you do *not* negate conditions in the middle of the statement (like "$|x| < \delta$" in the example above), because this would correspond to changing the rules of the game, rather than just changing the role with the Demon. Moreover, be careful when negating inequalities: either the original statement or its negation (but not both of them) should include the equality case!

Alternatively, we can also only use the rules that the negation of $\forall x : E$ (where E is any mathematical expression) is $\neg(\forall x : E) \Leftrightarrow \exists x : \neg E$ and the negation of $\exists x : E$ is $\forall x : \neg E$. Carrying this out step by step, again for the example (d), we find:

$$\begin{aligned}
&\neg(\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}, |x| < \delta\ :\ x^2 < \varepsilon) \\
\Leftrightarrow\ &\exists \varepsilon > 0\ \neg(\exists \delta > 0\ \forall x \in \mathbb{R}, |x| < \delta\ :\ x^2 < \varepsilon) \\
\Leftrightarrow\ &\exists \varepsilon > 0\ \forall \delta > 0\ \neg(\forall x \in \mathbb{R}, |x| < \delta\ :\ x^2 < \varepsilon) \\
\Leftrightarrow\ &\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in \mathbb{R}, |x| < \delta\ :\ \neg(x^2 < \varepsilon) \\
\Leftrightarrow\ &\exists \varepsilon > 0\ \forall \delta > 0\ \exists x \in \mathbb{R}, |x| < \delta\ :\ x^2 \geq \varepsilon.
\end{aligned}$$

3. For each of the statements (a)-(e), write down two *trial games* for the corresponding Demon game: one in which we win, one in which the Demon wins.

   (That is, write down two sequences of legal moves and state who wins if those are the moves played. You are *not* asked to find a winning strategy.)

Solution. Here are some possible trial games – of course your games will be different.

(a)

	Trial Game 1	Trial Game 2
Demon picks $n =$	6	9
We pick $m =$	5	3
Who wins?	Demon	Us

(b)

	Trial Game 1	Trial Game 2
Demon picks $a =$	4	4
Demon picks $\varepsilon =$	0.5	0.5
We pick $\delta =$	1.5	0.1
Demon picks $x =$	5	4.05
Who wins?	Demon	Us

4. For each of (b)–(e), give a *winning strategy* for the corresponding Demon game. Briefly explain why the strategy works. Then write down this winning strategy as a mathematical proof. Here are some possible winning strategies:

   (b) Suppose that Demon picks $a \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$. We then choose $\delta = \varepsilon/3$. Then the Demon picks $x \in \mathbb{R}$ with $|x-a| < \delta$. We now have to check that we have won, i.e. that the statement after the colon is satisfied. We have

   $$|3x - 3a| = 3|x-a| < 3\delta = \varepsilon$$

   so indeed $|3x - 3a| < \varepsilon$ as required. We explain how to turn the above game into a mathematical proof; the cases (c)-(e) are similar.
   Proof. Given $a \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, let $\delta = \varepsilon/3$. Then for all $x \in \mathbb{R}$ with $|x-a| < \delta$ we have $|3x-3a| = 3|x-a| < 3\delta = \varepsilon$. □

   (c) One winning strategy is the following:

   Suppose the Demon picks $x \in \mathbb{Q}$ and then $y \in \mathbb{Q}$ with $y \neq x$.
   We pick $z = (x+y)/2$. Note that $z \in \mathbb{Q}$.
   There are two cases. If $x < y$ then $x = (x+x)/2 < (x+y)/2 = z = (x+y)/2 < (y+y)/2 = y$, so we win. The case $y < x$ is symmetric.

   (d) Perhaps the most direct approach is the following. Whatever value of ε the Demon gives us, we respond by choosing $\delta = \sqrt{\varepsilon}$. Here is the justification that this is a winning strategy.

   The Demon picks $\varepsilon > 0$.
   We pick $\delta = \sqrt{\varepsilon}$.
   Now the Demon picks $x \in \mathbb{R}$ with $|x| < \delta$.
   We win since $x^2 = |x|^2 < \delta^2 = \varepsilon$.

   We will see many Demon Games of this type once we work with continuous functions. In fact, we will later in this course see, that the given quantifier expression means that the function $f(x) = x^2$ is continuous at zero.

   (e) One possible winning strategy is

   Suppose the Demon picks $N \in \mathbb{N}$.
   We pick $n = N!$, noting that $n \geq N$.
   We pick $m = N$.
   Then $n = N! = m!$.

5. **Challenge.** Find a *winning strategy* for the Demon Game corresponding to the expression:

$$\forall x \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in \mathbb{R}, |x-y| < \delta\ :\ |x^2 - y^2| < \varepsilon.$$

Solution. This is quite difficult at the present stage, but you will later in this course learn some tricks to solve questions of this type! I only give some hints, but not a complete solution at this stage.

The Demon Game corresponding to the expression

$$\forall x \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in \mathbb{R}, |x - y| < \delta\ :\ |x^2 - y^2| < \varepsilon$$

is the following:

First, the Demon picks $x \in \mathbb{R}$.
Then the Demon picks $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$.
Then we pick $\delta \in \mathbb{R}$ with $\delta > 0$.
Then the Demon picks $y \in \mathbb{R}$ with $|x - y| < \delta$.
We win if $|x^2 - y^2| < \varepsilon$.

Rough work. The challenge is to choose $\delta > 0$ such that $|x^2 - y^2| < \varepsilon$. Our goal can be re-expressed as $|x + y| \times |x - y| < \varepsilon$. Note that

$$\begin{aligned} &|x + y| \le |x| + |y| \le |x| + |x| + \delta = 2|x| + \delta, \text{ and} \\ &|x - y| < \delta. \end{aligned}$$

So by choosing $\delta \le 1$ we may ensure $|x + y| \le 2|x| + 1$. And by choosing $\delta \le \varepsilon/(2|x| + 1)$ we may ensure $|x - y| < \varepsilon/(2|x| + 1)$. We see that it will all work! So now let's write that down properly.

Proof. Given $x \in \mathbb{R}$ and $\varepsilon > 0$, choose $\delta = \min\left\{\varepsilon/(2|x| + 1), 1\right\}$. Then

$$\begin{aligned} &|x + y| \le |x| + |y| \le |x| + |x| + \delta = 2|x| + \delta \le 2|x| + 1, \text{ and} \\ &|x - y| < \delta \le \varepsilon/(2|x| + 1). \end{aligned}$$

Hence

$$|x^2 - y^2| = |x + y| \times |x - y| < (2|x| + 1) \times \varepsilon/(2|x| + 1) = \varepsilon. \quad \square$$

Given our rough work, there is nothing mystical in our choice of δ!