Queen Mary
University of London

MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 0 (Sets and Logic)

1. Let $S = \{2k : k \in \mathbb{N}\}$ and $T = \{2k+1 : k \in \mathbb{N}\}$.

   (i) Compute $S \cup T$ and $S \cap T$.

   (ii) Is $\mathbb{N}_0 \subseteq S \cup T$? Justify your answer.

   (iii) Prove that $\forall x \in S : x^2 \in S$.

   (iv) Does (iii) hold with S replaced by T? Justify your answer with a short proof or counterexample.

   Solution. Before getting started, it is helpful to write out a few elements of each set:

   $$S = \{2, 4, 6, 8, 10, \ldots\}, \qquad T = \{3, 5, 7, 9, 11, \ldots\}.$$

   (a) We directly compute:

   $$S \cup T = \{2, 3, 4, 5, 6, \ldots\} = \{x \in \mathbb{N} : x \geq 2\},$$
   $$S \cap T = \emptyset.$$

   (b) We do not have $\mathbb{N}_0 \subseteq S \cup T$ because $0 \in \mathbb{N}_0$ whilst $0 \notin S \cup T$.

   (c) Given $x \in S$ we can write $x = 2k$ for some $k \in \mathbb{N}$. Then

   $$x^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

   Set $k' = 2k^2 \in \mathbb{N}$. Then $x^2 = 2k'$ and so $x^2 \in S$ as required.

   (d) The corresponding claim does hold for T. Given $x \in T$ we can write $x = 2k+1$ for some $k \in \mathbb{N}$. Then:

   $$x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

   Set $k' = 2k^2 + 2k \in \mathbb{N}$. Then $x^2 = 2k' + 1$ and so $x^2 \in T$ as required.

2. (Liebeck.) Consider the set

   $$A = \{\alpha, \{1, \alpha\}, \{3\}, \{\{1, 3\}\}, 3\}.$$

   Which of the following statements are true and which are false?

(a) $\alpha \in A$.
(b) $\{\alpha\} \notin A$.
(c) $\{1, \alpha\} \subseteq A$.
(d) $\{3, \{3\}\} \subseteq A$.
(e) $\{1, 3\} \in A$.
(f) $\{\{1, 3\}\} \subseteq A$.
(g) $\{\{1, \alpha\}\} \subseteq A$.
(h) $\{1, \alpha\} \notin A$.
(i) $\emptyset \subseteq A$.

Solution.

(a) True.
(b) True.
(c) False.
(d) True.
(e) False.
(f) False.
(g) True.
(h) False.
(i) True.

3. Let A, B, C be three sets with $A, B \subseteq C$. Prove that

(i) $A \subseteq B \Rightarrow (C \setminus B) \subseteq (C \setminus A)$;
(ii) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$;
(iii) $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$.

Important: to prove that two sets X and Y are equal you need to prove that $X \subseteq Y$ and $Y \subseteq X$.

Solution.

(i) To prove $X \subseteq Y$ we must prove that $x \in X \Rightarrow x \in Y$. Suppose therefore that $A \subseteq B$ and let $x \in C \setminus B$. This means that $x \in C$ and $x \notin B$.

We claim that $x \notin A$. Indeed suppose for a contradiction that $x \in A$. Since $A \subseteq B$ it follows that $x \in B$. But this is a contradiction.

Since $x \in C$ and $x \notin A$ we conclude that $x \in C \setminus A$. This completes the proof that $C \setminus B \subseteq C \setminus A$.

(ii) We first show that

$$C \setminus (A \cup B) \subseteq (C \setminus A) \cap (C \setminus B).$$

Let $x \in C \setminus (A \cup B)$. This means that $x \in C$ and $x \notin A \cup B$. Since $x \notin A \cup B$ it follows that $x \notin A$ and $x \notin B$. Then:

$$x \in C, x \notin A \Rightarrow x \in C \setminus A,$$
$$x \in C, x \notin B \Rightarrow x \in C \setminus B.$$

We conclude that $x \in (C \setminus A) \cap (C \setminus B)$ as required. It remains to show the reverse inclusion, namely:

$$(C \setminus A) \cap (C \setminus B) \subseteq C \setminus (A \cup B).$$

Let $x \in (C \setminus A) \cap (C \setminus B)$. This means that $x \in C \setminus A$ and $x \in C \setminus B$. We obtain:

$$x \in C \setminus A \Rightarrow x \in C, x \notin A$$
$$x \in C \setminus B \Rightarrow x \in C, x \notin B.$$

Since $x \notin A$ and $x \notin B$, we see that $x \notin A \cup B$. Together with $x \in C$ this implies $x \in C \setminus (A \cup B)$ as required.

(iii) This is similar to the previous part. It is better for you to work out the details yourself than to have me write them down here. Try following the steps of the previous part exactly. If you are struggling, let me know.

4. (Liebeck.) Which of the following arguments are valid? For the valid ones, write down the argument symbolically.

(a) I eat chocolate if I am sad. I am not sad. Therefore I am not eating chocolate.

(b) I eat chocolate only if I am sad. I am not sad. Therefore I am not eating chocolate.

(c) If a movie is not worth seeing, then it was not made in England. A movie is worth seeing only if critic Ivor Smallbrain reviews it. The movie “Cat on a Hot Tin Proof” was not reviewed by Ivor Smallbrain. Therefore “Cat on a Hot Tin Proof” was not made in England.

Solution.

(a) Invalid. We are given

$$\text{I am sad} \Rightarrow \text{I eat chocolate.}$$

But this does not allow us to conclude anything about what happens when I am not sad. In particular, the above is *not* the same as

$$\text{I am not sad} \Rightarrow \text{I do not eat chocolate.}$$

(b) Valid. We are told that

$$\text{I eat chocolate} \Rightarrow \text{I am depressed.}$$

The contrapositive thus gives:

$$\text{I am not depressed} \Rightarrow \text{I do not eat chocolate.}$$

(c) Valid. We are told that

$$\text{Not worth seeing} \Rightarrow \text{not made in England}$$
$$\text{Worth seeing} \Rightarrow \text{Ivor Smallbrain reviews it}$$

The contrapositive of the second statement gives:

$$\text{Ivor Smallbrain did not review it} \Rightarrow \text{not worth seeing}$$

Thus, if "Cat on a Hot Tin Proof" was not reviewed by Ivor Smallbrain, it is not worth seeing, and therefore it was not made in England.

5. (Liebeck.) Which of the following statements are true, and which are false?

(a) $n = 3 \Rightarrow n^2 - 2n - 3 = 0$.
(b) $n^2 - 2n - 3 = 0 \Rightarrow n = 3$.
(c) $\forall a, b \in \mathbb{Z}, ab$ is a perfect square $\Rightarrow a$ and b are perfect squares.
(d) $\forall a, b \in \mathbb{Z}, a$ and b are perfect squares $\Rightarrow ab$ is a perfect square.

Solution.

(a) True, since $3^2 - 2 \cdot 3 - 3 = 0$.
(b) False, since $n = -1$ is also a solution.
(c) False: take $a = b = 2$.
(d) True: if $a = k^2$ and $b = l^2$ then $ab = k^2 l^2 = (kl)^2$.

6. Prove that there exists a unique $x \in \mathbb{N}$ such that $x^2 = x$.

Hint: Start by proving $\exists x \in \mathbb{N} : x^2 = x$. Then prove the uniqueness of x by contradiction.

Solution. Setting $x = 1 \in \mathbb{N}$ we have $x^2 = 1^2 = 1 = x$. We conclude that there exists a natural number x such that $x^2 = x$. We must now show that this natural number is unique.

Suppose for a contradiction that there exists a natural number $y \in \mathbb{N}$ with $y \neq 1$ and $y^2 = y$. We rearrange the expression to obtain:

$$y(y - 1) = 0.$$

It follows that $y = 0$ or $y = 1$. Since $y \in \mathbb{N} = \{1, 2, 3, \ldots\}$ we cannot have $y = 0$. On the other hand since $y \neq 1$ we cannot have $y = 1$. We arrive at a contradiction. $\square$