WEEK 4 NOTES

1. The wave equation on the real line

We continue the theory wave equation

$$U_{tt} = c^2 U_{xx}, \qquad c \equiv \sqrt{\frac{F}{\rho}}. \tag{1.1}$$

on the real line so that there are no boundary conditions —physically, this means that we consider an infinitely long vibrating string. This is a useful *idealisation*.

1.1. Solution in terms of initial conditions.

Now, suppose one has the initial conditions

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x).$$

One needs to two initial conditions as the equation is second order. At $t = 0$ the general solution deduced last week gives

$$U(x,0) = G(x) + F(x) = f(x). \tag{1.2}$$

Moreover, a direct computation using the chain rule gives

$$U_t(x,t) = cF'(x+ct) - cG'(x-ct),$$

so that

$$U_t(x,0) = cF'(x) - cG'(x) = g(x) \tag{1.3}$$

Differentiating (1.2) with respect to x one obtains the system of equations

$$\begin{aligned} f'(x) &= G'(x) + F'(x), \\ g(x) &= cF'(x) - cG'(x). \end{aligned}$$

Adding and subtracting these equations one finds that

$$\begin{aligned} F'(x) &= \frac{1}{2c}\big(g(x) + cf'(x)\big), \\ G'(x) &= \frac{1}{2c}\big(cf'(x) - g(x)\big). \end{aligned}$$

Integrating the first of these equations with respect to x one finds that

$$\begin{aligned} F(x) - F(0) &= \int_0^x \frac{1}{2c}\big(g(s) + cf'(s)\big)ds \\ &= \frac{1}{2}\big(f(x) - f(0)\big) + \frac{1}{2c}\int_0^x g(s)ds, \end{aligned}$$

where in the second line we have used the *Fundamental theorem of Calculus*. Moreover, using this last expression one has that

$$\begin{aligned} G(x) &= f(x) - F(x) \\ &= f(x) - \frac{1}{2}f(x) + \frac{1}{2}f(0) - \frac{1}{2c}\int_0^x g(s)ds - F(0) \\ &= \frac{1}{2}f(x) + \frac{1}{2}f(0) - \frac{1}{2c}\int_0^x g(s)ds - F(0). \end{aligned}$$

It follows then that

$$\begin{aligned} U(x,t) &= G(x-ct) + F(x+ct) \\ &= \frac{1}{2}f(x-ct) + \frac{1}{2}f(0) - \frac{1}{2c}\int_0^{x-ct} g(s)ds - F(0) \\ &\quad + \frac{1}{2}f(x+ct) - \frac{1}{2}f(0) + \frac{1}{2c}\int_0^{x+ct} g(s)ds + F(0). \end{aligned}$$

Simplifying and rearranging one obtains the expression

$$U(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds, \tag{1.4}$$

which is known as *D'Alembert's solution*.

Note. *Observe that by prescribing initial conditions one obtains a unique solution.*

Example 1.1. The wave equation $U_{tt} = c^2 U_{xx}$ with initial position $U(x,0) = \sin x$ and initial velocity $U_t(x,0) = 0$ is

$$\begin{aligned} U(x,t) =& \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] \\ =& \sin x \cos(ct). \end{aligned}$$

1.2. **Where does the change of variables come from?** To explain the change of variables

$$u = x - ct, \qquad v = x + ct. \tag{1.5}$$

one observes that the wave equation can be rewritten as

$$U_{tt} - c^2 U_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) U = 0.$$

Letting

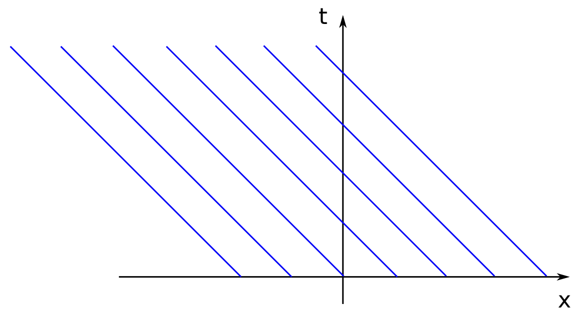
$$W \equiv \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) U,$$

then

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) W = \frac{\partial W}{\partial t} - c\frac{\partial W}{\partial x} = W_t - cW_x = 0.$$

Thus, W satisfies a first order pde with constant coefficients —we have already studied the solutions to this equation. The characteristics are lines with negative slope $dt/dx = -1/c$ (negative slope) so that

$$x + ct = \text{constant}.$$

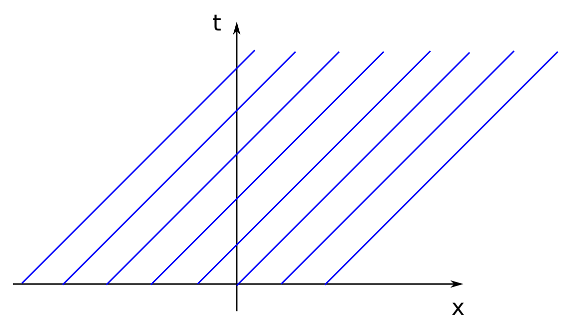


Once we know W one has to solve the equation

$$U_t + cU_x = W$$

which is, again, a first order pde with constant coefficients —observe, however, that the equation is inhomogeneous. The slope of the characteristics is $dt/dx = 1/c$ (positive slope) so that

$$x - ct = \text{constant}.$$

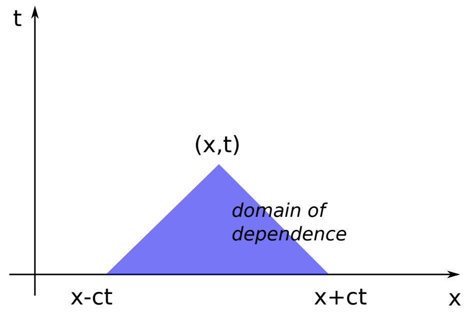


Note. *Thus, the wave equation has two sets of characteristics —that is, there is information travelling in two directions: to the left and to the right.*

1.3. **Interpretation of D'Alembert's solution.** Formula (1.4) can be read as saying

$$U(x,t) = \big(\text{average of } U(x,0) \text{ on } x - ct \text{ and } x + ct\big) \\ + \big(\text{average of } U_t(x,0) \text{ over the interval } [x-ct, x+ct]\big).$$

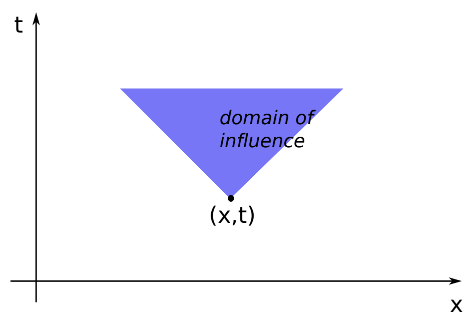
Hence, $U(x,t)$ only depends on the initial conditions on the interval $[x - ct, x + ct]$ —see the figure below.



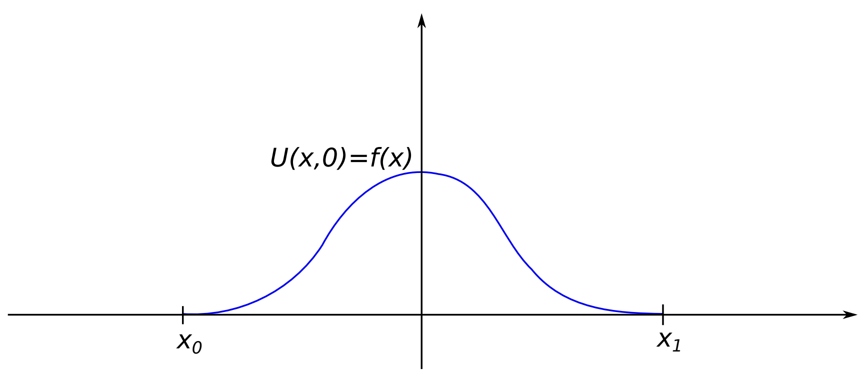
The region in the (x,t) diagram that have an influence in the value of $U(x,t)$ at (x,t) is called the **domain of dependence** of (x,t).

Note. *This has connections with Relativity (MTH6132) —information cannot travel at infinite speed.*

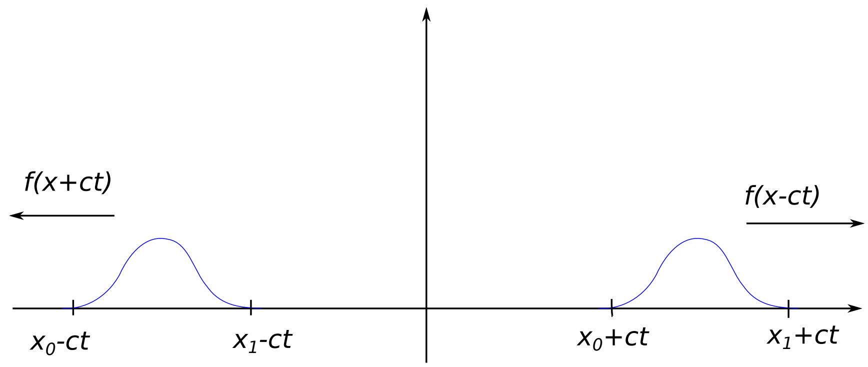
Conversely, given a point (x,t) (event) it influences the region shown below:



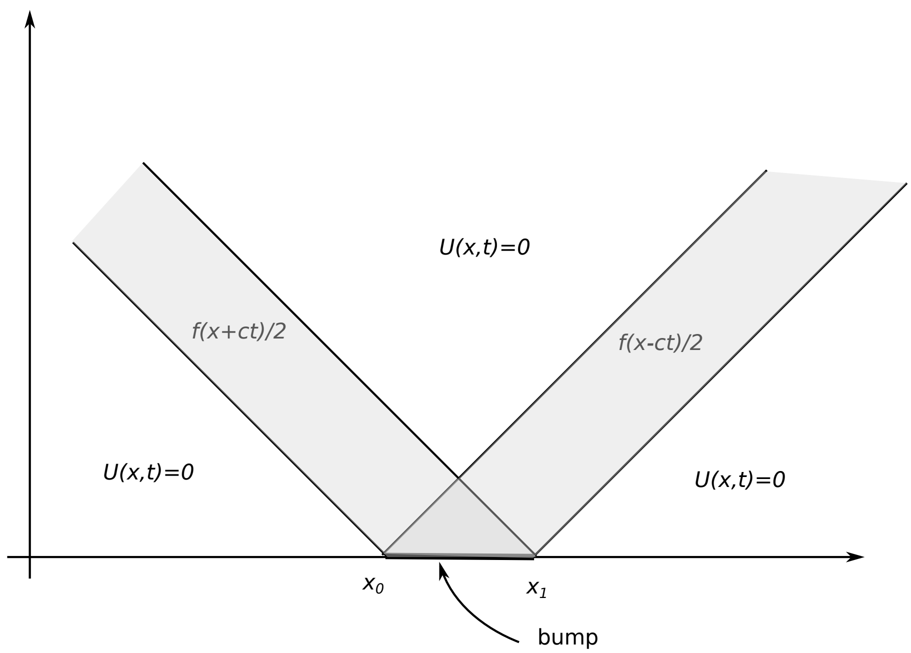
This region is called the **domain of influence**.
Also, assume that $g(x) = 0$ and that $f(x)$ has the shape of a bump:



Then, at later times the solution looks like:



That is, one has two bumps, half the size of the initial one moving in opposite directions. The above situation can be described in a diagram in the (x,t) plane as follows:



2. Some invariant properties of wave equations on the real line

From a given solution $U(x,t)$ to the wave equation (1.1) on the real line $x \in \mathbb{R}$, we can construct new solutions to the equation

Proposition 2.1. *If $U(x,t)$ is a solution to the wave equation* (1.1) *on the real line, so are*

$$V(x,t) = U(\alpha x, \alpha t), \textit{for any } \alpha \in \mathbb{R},$$
$$W(x,t) = U(x,-t).$$

Proposition 2.2. *Let $U_1(x,t)$ solves the the advection equation $(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})U_1 = 0$ on the real line,* *and $U_2(x,t)$ solves the the advection equation $(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})U_2 = 0$ on the real line. Then both U_1 and U_2 solves the wave equation* (1.1) *on the real line.*

We will leave it as an exercise to show these 2 propositions. Problem sets also contains some similar questions about the invariant properties.

3. Conservation of energy

Consider the wave equation on the line:

$$\begin{aligned} &U_{tt} = c^2 U_{xx}, \qquad x \in \mathbb{R} \\ &U(x,0) = f(x), \qquad U_t(x,0) = g(x), \end{aligned}$$

where $f(x),\ g(x) = 0$ for $|x| > R$, with R some big number. This means that $f(x)$ and $g(x)$ vanish for large $|x|$ —functions of this type are said to have *compact support*.

3.1. **Derivation.** Multiply now the wave equation by U_t on both sides:

$$U_t U_{tt} = c^2 U_{xx} U_t.$$

Observing that

$$U_t U_{tt} = \frac{1}{2}\frac{\partial}{\partial t}\left(U_t^2\right),$$

one has then that

$$\frac{1}{2}\frac{\partial}{\partial t}\left(U_t^2\right) - c^2 U_{xx} U_t = 0.$$

Integrating over the real line one then gets that

$$\begin{aligned} 0 &= \int_{\infty}^{-\infty} \left(\frac{1}{2}\frac{\partial}{\partial t}\left(U_t^2\right) - c^2 U_{xx} U_t\right) dx \\ &= \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2} U_t^2 dx - c^2 \int_{-\infty}^{\infty} U_{xx} U_t dx \\ &= \frac{d}{dt}\left(\int_{-\infty}^{\infty} \frac{1}{2} U_t^2 dx\right) - \left(U_t U_x \Big|_{-\infty}^{\infty} - c^2 \int_{-\infty}^{\infty} U_x U_{xt} dx\right) \\ &= \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2} U_t^2 dx + c^2 \int_{-\infty}^{\infty} U_x U_{xt} dx, \end{aligned}$$

where to pass from the second to the third line we have used integration by parts and in the third line that $U(x,t) = 0$ if $|x| \to \infty$. Finally, observing that

$$U_x U_{xt} = \frac{1}{2}\frac{\partial}{\partial t}(U_x^2),$$

one concludes that

$$0 = \frac{d}{dt}\int_{-\infty}^{\infty} \frac{1}{2} U_t^2 dx + c^2 \int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial t}(U_x^2) dx,$$

so that

$$\frac{d}{dt}\left(\frac{1}{2}\int_{-\infty}^{\infty} \left(U_t^2 + c^2 U_x^2\right) dx\right) = 0.$$

In other words, the quantity in brackets is constant in time. This calculation suggests the following definition:

Definition 3.1. The **energy** $E[U](t)$ of a solution to the wave equation is given by

$$E[U](t) \equiv \frac{1}{2}\int_{-\infty}^{\infty} \big(U_t^2 + c^2 U_x^2\big) dx.$$

Hence, the previous calculations show that

$$\frac{d}{dt}E[U](t) = 0,$$

that is, the energy is conserved —i.e. independent of t (**law of conservation of total energy**). The term $\int U_t^2/2$ is called the **kinetic energy** and $\int \frac{c^2}{2} U_x^2$ the **potential energy**.

3.2. **An application: uniqueness of solutions.** In this subsection we show how the total energy can be used to show that a solution to the initial value problem

$$U_{tt} - c^2 U_{xx} = 0, \qquad x \in \mathbb{R}$$
$$U(x,0) = f(x), \qquad U_t(x,0) = g(x)$$

if it exists, then it must be unique.

Suppose one has 2 solutions U_1 and U_2 an let $W \equiv U_1 - U_2$. As the wave equation is linear one has that

$$W_{tt} - c^2 W_{xx} = 0,$$
$$W(x,0) = 0, \qquad W_t(x,0) = 0.$$

The energy of W can be directly computed to be

$$\begin{aligned} E[W](t) &= E[W](0) \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \big(W_t^2(x,0) + c^2 W_x^2(x,0)\big) dx, \\ &= 0. \end{aligned}$$

This means, in particular, that

$$\int_{-\infty}^{\infty} \big(W_t^2(x,t) + c^2 W_x^2(x,t)\big) dx = 0,$$

but $W_t^2 \geq 0$, $W_x^2 \geq 0$ so that, in order for the integral to vanish one actually needs

$$W_t(x,t) = 0, \qquad W_x(x,t) = 0.$$

Thus $W(x,t)$ is constant for all x, t. But $W(x,0) = 0$ so that $W(x,t) = 0$. Hence, $U_1 = U_2$ —that is, the solution is unique.