WEEK 3 NOTES

1. Variable coefficients first order linear PDEs (continued)

In the past week, we have seen the general theory for solving the equations

$$a(x,y)U_x + b(x,y)U_y = c(x,y)U + d(x,y),$$

and have seen some examples with the right hand side of the equations being zero.

Notice that when the right hand side is zero, the solution U are constant along the characteristic curves.

Let's begin this week by more general examples.

Example 1.1. Find the general solution to

$$(1+x^2)U_x + U_y = 0.$$

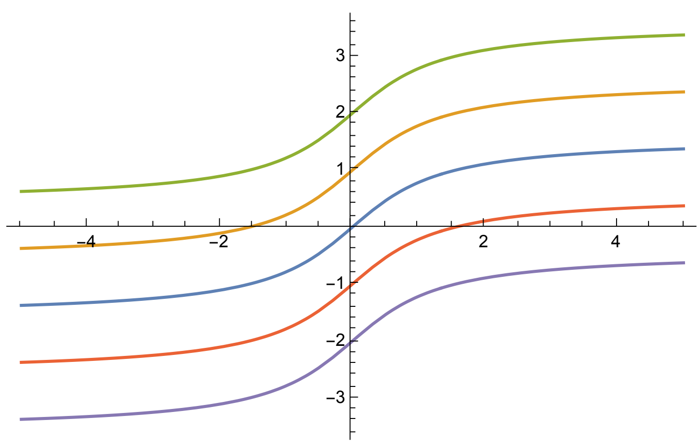
In this case the equation for the characteristic curves is given by

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

The solution to this ode is given by (why?):

$$y(x) = \arctan x + C, \qquad C \quad \text{a constant.}$$

A plot of the characteristics for various values of C is given below.



Again one can check that they actually cover the whole plane.

Now, from the general theory one has that

$$U(x, \arctan x + C)$$

is constant along the characteristics —of course, one can also verify it by direct computation. Hence,

$$U(x, \arctan x + C) = f(C) = \text{constant for given } C.$$

On the other hand one has that

$$C = y - \arctan x$$

so that

$$U(x,y) = U(0, y - \arctan x) = f(y - \arctan x),$$

with f a function of a single argument. This is the general solution of the equation.

Example 1.2. Find the general solution to

$$U_x + 2xy^2 U_y = 0.$$

In this case the equation for the characteristics is given by

$$\frac{dy}{dx} = 2xy^2.$$

It follows that

$$\int \frac{dy}{y^2} = \int 2x dx + C.$$

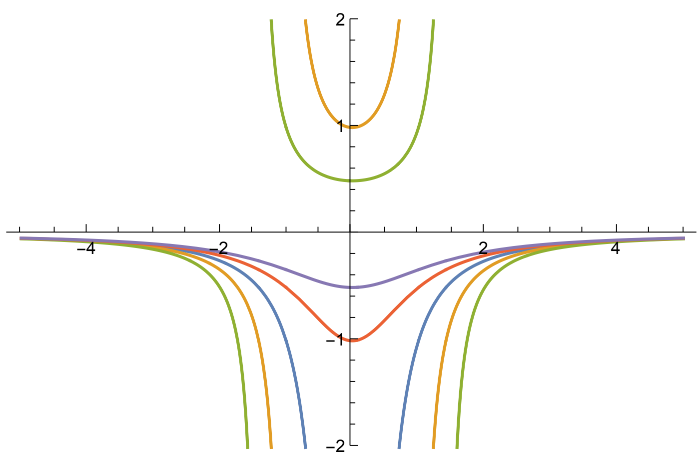
Integrating one gets

$$-\frac{1}{y} = x^2 + C,$$

so that after some reorganisation one ends up with

$$y = \frac{1}{C - x^2}.$$

A plot of the characteristic curves for various choices of C are given below.



Note that the curves do not seem to fill the plane so that the solution may not exist for all (x,y). Again, from general theory we know that $U(x,y)$ is constant along these curves. That is,

$$U(x, y(x)) = f(C).$$

Observing that in this case

$$C = x^2 + \frac{1}{y}$$

one concludes that the required general solution is given by

$$U(x,y) = f\left(x^2 + \frac{1}{y}\right).$$

Example 1.3. Solve the initial value problem

$$\begin{cases} U_t + xU_x = \sin t, x \geq 0 \\ U(x,0) = x \end{cases}$$

This is an example of an inhomogeneous equation. The ode for the characteristics is in this case given by

$$\frac{dt}{dx} = \frac{1}{x}.$$

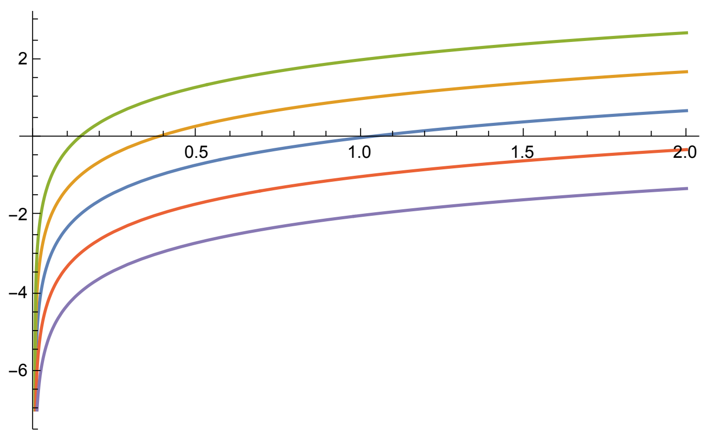
The general solution to this ODE is given by

$$t(x) = \ln x + \tilde{C},$$

or

$$e^t = Cx.$$

It will be convenient to rewrite the latter in a slightly different form: $t = \ln x + \ln C$, so that $t = \ln Cx$. A plot of the curves for various values of C is given below:



From the general theory (or direct computation) one further obtains the ode

$$\frac{d}{dx}U(x, t(x)) = \frac{\sin t}{x}$$

Expressing t in terms of x using the equation for the characteristic curves one finally finds that

$$\frac{dU}{dx} = \frac{\sin\ln(Cx)}{x}.$$

Using the substitution $z = \ln Cx$ one has that

$$\int \frac{\sin\ln(Cx)}{x}dx = \int \sin z dz = -cosz = -\cos\ln Cx,$$

so that

$$U = -\cos\ln Cx + f(C).$$

In particular, using the characterisitic equation $e^t = cx$ to replace C by $C = \frac{e^t}{x}$, we get the general solution

$$U(x,t) = -\cos\ln(e^t) + f(\frac{e^t}{x}) = -\cos t + f(\frac{e^t}{x}).$$

To specify the function f, we use the initial condition. When $t = 0$, we have $Cx = 1$ and $x = \frac{1}{C}$, so

$$\frac{1}{C} = x = U(x, 0) = -\cos\ln(CX) + f(C) = -1 + f(C).$$

This gives $f(C) = 1 + \frac{1}{C}$. Substituting $e^t = Cx$ general solution to the PDE is

$$U(x, t) = -\cos\ln Cx + f(C) = -\cos t + 1 + \frac{x}{e^t}$$

Example 1.4. Find the general solution to the equation

$$xU_x + yU_y = kU, \qquad k \quad \text{a constant.}$$

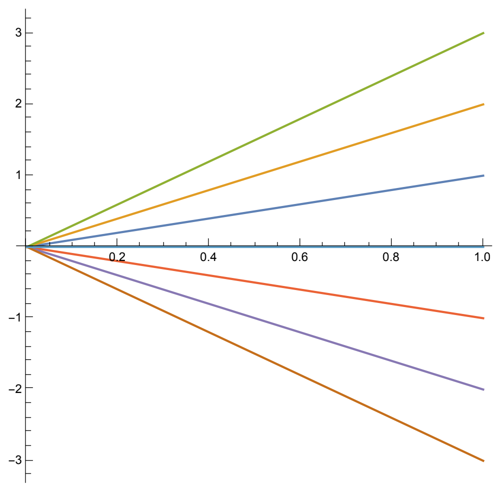
This equation is known as an Euler equation. The characteristic equation is then given by

$$\frac{dy}{dx} = \frac{y}{x},$$

which has general solution given by

$$y(x) = Cx,$$

with C a constant —why? A plot of the curves is shown below —observe that they intersect at the origin.



From the general theory (or direct computation) one has that

$$\frac{d}{dx}U(x, y(x)) = \frac{k}{x}U.$$

We can then integrate it as follows:

$$\int \frac{dU}{U} = k\int \frac{dx}{x} + f(C)$$

so that

$$U(x, y(x)) = f(C)x^k.$$

Now, using the equation $C = y/x$ to eliminate C one obtains the general solution

$$U(x, y) = f\left(\frac{y}{x}\right)x^k.$$

2. A SIMPLE NON-LINEAR FIRST ORDER PDES

In this section, we will try to use the method of characteristic to solve a simple non-linear first order PDEs whose characteristics are straight lines.

Example 2.1. Solve the Initial value problem

$$\begin{cases} U_x + U_t + U^2 = 0 \\ U(x,0) = \sin x \end{cases} \tag{2.1}$$

This PDE is **non-linear** and **homogenous**!

To solve this problem, we first move the non-linear term to the write hand side.

$$U_x + U_t = -U^2.$$

Next, using the the same theory as finding characteristics as in 2.1 in Week 2 notes, we get the characteristic equations

$$\frac{\partial t}{\partial x} = \frac{1}{1} = 1.$$

And thus the characteristics are straight lines

$$x = t + C$$

So the equation (2.1) becomes

$$\frac{d}{dx}U(x,t(x)) = -U^2$$
$$\frac{-1}{U^2(x,t(x))}\frac{d}{dx}U(x,t(x)) = 1.$$

Integrating both sides, we get

$$\frac{1}{U(x,t(x))} = x + f(C).$$

Here f can be any differentiable functions.

Now using the initial value $U(x,0) = \sin x$ to specify. When $t = 0$, we have $x = C$. And thus when $t = 0$, we have

$$\frac{1}{C + f(C)} = \frac{1}{x + f(C)} = U(x,0) = \sin x = \sin C.$$

So

$$f(C) = \frac{1}{\sin C} - C.$$

Using $C = x - t$, we then have

$$x + f(C) = x + \frac{1}{\sin C} - C = x + \frac{1}{\sin(x-t)} - (x-t) = t + \frac{1}{\sin(x-t)}.$$

So the general solution to the PDE is

$$U(x,t) = \frac{1}{t + \frac{1}{\sin(x-t)}} = \frac{\sin(x-t)}{t\sin(x-t)+1}.$$

The solution exists when $t\sin(x-t) + 1 \neq 0$.

3. SECOND ODER PARTIAL DIFFERENTIAL EQUATIONS

In this section we briefly look at the classification of second order partial differential equations with constant coefficients.

3.1. Introduction. The most general second order partial differential equation with constant coefficients is given by

$$aU_{xx} + 2bU_{xy} + cU_{yy} + dU_x + eU_y + fU = h(x,y) \tag{3.1}$$

with

$$a, \quad b, \quad c, \quad d, \quad e, \quad f,$$

are constants and $h(x,y)$ is an arbitrary function. The terms with the highest order derivatives, namely

$$aU_{xx} + 2bU_{xy} + cU_{yy} \tag{3.2}$$

are called the *principal part*. It determines the character of the solutions of the equation. In the following, to avoid messy computations, we consider only the principal part —i.e. we set d, e and f to zero. Particular cases of equation (3.1) are

$$\begin{aligned} &U_{xx} - U_{tt} = 0 && \text{(wave equation)},\\ &U_{xx} + U_{yy} = 0 && \text{(Laplace equation)},\\ &U_{xx} - U_t = 0 && \text{(heat equation)}. \end{aligned}$$

The solutions to each of these equations have a completely different behaviour. In the following, we will see that, in a sense, these are the only possibilities.

3.2. Quadratic forms. The basic observation is the following: compare the principal part (3.2) with the *quadratic form*

$$ax^2 + 2bxy + cy^2.$$

We know from basic geometry that the solutions to the equation defined by this quadratic form represents a conic section —i.e. a hyperbola, a parabola or an ellipse. The type of conic section depends on the coefficients in the quadratic form. More precisely, completing squares one has that

$$ax^2 + 2bxy + cy^2 = a\left(\left(x + \frac{b}{a}y\right)^2 + \left(\frac{ac-b^2}{a^2}\right)y^2\right).$$

One then has the following classification:

$$\begin{aligned} &b^2 - ac > 0 && \text{hyperbola},\\ &b^2 - ac = 0 && \text{parabola},\\ &b^2 - ac < 0 && \text{ellipse}. \end{aligned}$$

One can do something similar with the principal part (3.2). One can readily check that

$$aU_{xx} + 2bU_{xy} + cU_{yy} = a\left(\left(\frac{\partial}{\partial x} + \frac{b}{a}\frac{\partial}{\partial y}\right)^2 + \left(\frac{ac-b^2}{a^2}\right)\frac{\partial}{\partial y^2}\right)U.$$

Accordingly, one classifies the pde's according to the same criteria as for the quadratic forms —more precisely, one says that (3.1) is

$$\begin{aligned} &b^2 - ac > 0 && \text{hyperbolic pde},\\ &b^2 - ac = 0 && \text{parabolic pde},\\ &b^2 - ac < 0 && \text{elliptic pde}. \end{aligned}$$

One can readily check that

$$\begin{array}{rl} \text{wave equation} & \text{hyperbolic,} \\ \text{Laplace equation} & \text{elliptic,} \\ \text{heat equation} & \text{parabolic.} \end{array}$$

3.3. **A change of variables.** Consider now new coordinates (x', y') given by

$$\begin{aligned} x' &= x, \\ y' &= -\frac{b}{a}x + y, \end{aligned}$$

so that

$$\begin{aligned} x &= x', \\ y &= y' + \frac{b}{a}x. \end{aligned}$$

Using the chain rule for partial derivatives one finds that

$$\begin{aligned} \frac{\partial}{\partial y'} &= \frac{\partial}{\partial y}, \\ \frac{\partial}{\partial x'} &= \frac{\partial}{\partial x} + \frac{b}{a}\frac{\partial}{\partial y}. \end{aligned}$$

Substituting the above into the principal part (3.2) a calculation readily gives

$$aU_{xx} + 2bU_{xy} + cU_{yy} = a\left(U_{x'x'} + \left(\frac{ac-b^2}{a^2}\right)U_{y'y'}\right).$$

Now, if $ac - b^2 < 0$ one can write

$$\begin{aligned} U_{x'x'} + \left(\frac{ac-b^2}{a^2}\right)U_{y'y'} &= U_{x'x'} - \frac{|ac-b^2|}{|a|^2}U_{y'y'} \\ &= \left(\frac{\partial}{\partial x'} + \frac{\sqrt{|ac-b^2|}}{|a|}\frac{\partial}{\partial y'}\right)\left(\frac{\partial}{\partial x'} - \frac{\sqrt{|ac-b^2|}}{|a|}\frac{\partial}{\partial y'}\right)U. \end{aligned}$$

In fact, one can eliminate the factor $\sqrt{|ac-b^2|}/|a|$ by a further change of variables.

Note. *The classification also works if the coefficients depend on the coordinates. In that case the character of the equation can change from point to point. As an example one has the equation*

$$U_{xx} + xU_{yy} = 0.$$

We will now focus on hyperbolic equations in the coming weeks. A typical example is the wave equations

4. The wave equation in $1+1$ dimension

The wave equation in $1+1$ dimension is

$$U_{tt} - c^2U_{xx} = 0 \tag{4.1}$$

with c a constant (**wave speed**) and $x \in I \subseteq \mathbb{R}$, $t > 0$ —i.e. I is an interval which can be finite, semi-infinite or infinite. The equation is supplemented by *initial conditions*

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x),$$

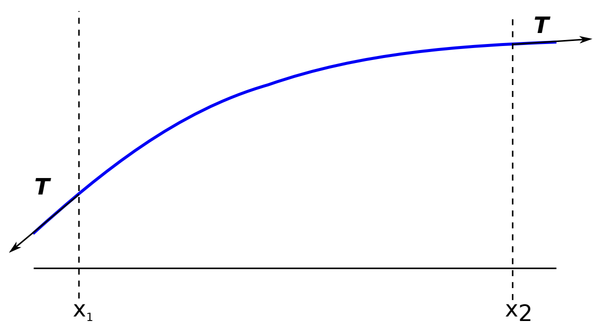
and, possibly, also *boundary conditions* if $I \neq \mathbb{R}$.

Note. *In* $3+1$ *dimensions the wave equation takes the form*

$$U_{tt} - c^2(U_{xx} + U_{yy} + U_{zz}) = 0.$$

The wave equation arises in problems describing the vibration of strings and membranes. More generally, the equations describe sound waves, electromagnetic waves, seismic waves, gravitational waves, propagation of epidemics, movement of populations, ...

4.1. **The vibrating string.** Consider, in the following, a *flexible, elastic, homogeneous string* of length L undergoing small transverse vibrations. Assume that the motion is restricted to a plane, and let $U(x,t)$ be the displacement from equilibrium position at time t and position x.



If the string is perfectly flexible, then the force (*tension*) responsible for the displacement is directed tangentially along the string and is constant in time since the string is homogeneous. The position of the string at a point x is then given by $\big(x, U(x,t)\big)$ and the slope of the string at x is that of the tangent. The tangent vector at a point x is given by

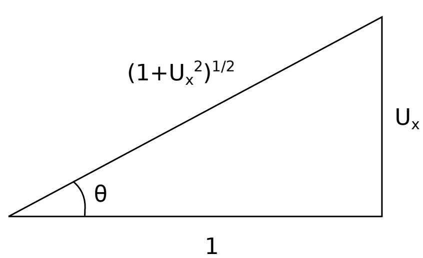
$$\frac{d}{dx}\big(x, U(x,t)\big) = \big(1, U_x(x,t)\big).$$

The key to obtaining an equation for $U(x,t)$ is **Newton's second law**

$$\vec{F} = m\vec{a}.$$

Now, from the diagram one has that

$$\cos\theta = \frac{1}{\sqrt{1+U_x^2}}, \qquad \sin\theta = \frac{U_x}{\sqrt{1+U_x^2}}.$$



The tension $\vec{F}$ is then given by

$$\vec{F} = F(x)\big(\cos\theta, \sin\theta\big) = \frac{F}{\sqrt{1+U_x^2}}\big(1, U_x\big),$$

where $F = F(x)$ is the norm of $\vec{F}$ and is assumed to be independent of time —see above. From the discussion above we have that the position of an element of string is then given by

$$\vec{x} = \big(x, U(x,t)\big)$$

so that its velocity and acceleration are given, respectively, by

$$\dot{\vec{x}} = \big(0, U_t\big), \qquad \ddot{\vec{x}} = \big(0, U_{tt}\big),$$

where the overdot $\dot{}$ denotes differentiation with respect to t. We can now compute the force along a segment of string $[x_1, x_2]$ using Newton's law:

$$\frac{F(x)}{\sqrt{1+U_x^2}}\big(1, U_x\big)\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} \rho \cdot \big(0, U_{tt}(s,t)\big)\mathrm{d}s,$$

where ρ is the density of the string (mass/unit length) which we assume to be constant. As this is a vector expression it implies two equations for the x and y components. The x component gives the equation

$$\frac{F(x)}{\sqrt{1+U_x^2}}\Big|_{x_1}^{x_2} = 0, \tag{4.2}$$

while the y component gives

$$\frac{F(x)}{\sqrt{1+U_x^2}}U_x\Big|_{x_1}^{x_2} = \int_{x_1}^{x_2} \rho U_{tt}(s,t)ds. \tag{4.3}$$

Now, as U_x is assumed to be small, then using Taylor series one has that

$$\sqrt{1+U_x^2} \approx 1 + \frac{1}{2}U_x^2 + \cdots \approx 1.$$

Using this approximation it follows from (4.2) that $F(x)$ is constant —i.e. independent of x. Equation (4.3) then gives

$$F\Big(U_x(x_2,t) - U_x(x_1,t)\Big) = \int_{x_1}^{x_2} \rho U_{tt}(s,t)ds.$$

Now, the fundamental theorem of calculus then gives that

$$F\big(U_x(x_2,t) - U_x(x_1,t)\big) = F\int_{x_1}^{x_2} U_{xx}(s,t)ds.$$

Hence,

$$F\int_{x_1}^{x_2} U_{xx}(x,t)ds = \int_{x_1}^{x_2} \rho U_{tt}(s,t)ds.$$

As the points x_1 and x_2 are arbitrary the integrands must be equal so that

$$\frac{F}{\rho}U_{xx} = U_{tt}.$$

We write the latter as

$$U_{tt} = c^2 U_{xx}, \qquad c \equiv \sqrt{\frac{F}{\rho}}. \tag{4.4}$$

This is the (homogeneous) *wave equation*. The constant c is called the **wave speed**.

4.2. **Computing the general solution.** Consider the change of variables

$$u = x - ct, \qquad v = x + ct. \tag{4.5}$$

Using the chain rule one has that

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \\
\frac{\partial}{\partial t} &= \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} = c\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right).
\end{aligned}$$

The second derivatives are computed as

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \left(\frac{\partial}{\partial x}\right)^2 = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right) \\
&= \frac{\partial^2}{\partial u^2} + 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial v^2}, \\
\frac{\partial^2}{\partial t^2} &= \left(\frac{\partial}{\partial t}\right)^2 = c^2\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right)\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right) \\
&= c^2\left(\frac{\partial^2}{\partial v^2} - 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial u^2}\right).
\end{aligned}$$

Thus, one has that

$$\begin{aligned}
U_{tt} - c^2 U_{xx} &= c^2\left(\frac{\partial^2}{\partial v^2} - 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial u^2}\right)U - c^2\left(\frac{\partial^2}{\partial u^2} + 2\frac{\partial^2}{\partial u \partial v} + \frac{\partial^2}{\partial v^2}\right)U \\
&= -4c^2\frac{\partial^2 U}{\partial u \partial v}.
\end{aligned}$$

Hence, we have transformed the original wave equation (4.4) into

$$\frac{\partial^2 U}{\partial u \partial v} = 0. \tag{4.6}$$

To solve equation (4.6) we notice that

$$\frac{\partial^2 U}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial U}{\partial v}\right),$$

so that integrating with respect to u one has

$$\int \frac{\partial}{\partial u}\left(\frac{\partial U}{\partial v}\right) du = \frac{\partial U}{\partial v} = f(v),$$

with $f(v)$ an arbitrary function of v. Integrating now with respect to v one gets

$$U(u, v) = \int f(v)dv + G(u),$$

with $G(u)$ an arbitrary function of u. Now, observe that the integral $\int f(v)dv$ is an arbitrary function of v so that one can write

$$U(u, v) = F(v) + G(u)$$

where $F(v)$ is another arbitrary function of v. Writing the latter in terms of the coordinates (x, y) one finds that

$$U(x, t) = G(x - ct) + F(x + ct). \tag{4.7}$$

This is the general solution to the wave equation (4.4).