PROBLEM SET 7 FOR MTH 6151

1. Solve

$$U_{xx} + U_{yy} = 0,$$

in the rectangle

$$\Omega = \{0 < x < a,\, 0 < y < b\}$$

with boundary conditions

$$U(0,y) = g_1(y), \qquad U(a,y) = 0, \qquad U(x,0) = 0, \qquad U(x,b) = 0.$$

2. Solve

$$U_{xx} + U_{yy} = 0,$$

in the rectangle

$$\Omega = \{0 < x < a,\, 0 < y < b\}$$

with boundary conditions

$$U(0,y) = 0, \qquad U(a,y) = 0, \qquad U(x,0) = 0, \qquad U(x,b) = f_2(x).$$

3. Find the harmonic function $U(x,y)$ in the square

$$\Omega = \{0 < x < 1,\, 0 < y < 1\}$$

with the boundary conditions

$$U(x,0) = x, \qquad U(x,1) = 0, \qquad U(0,y) = 0, \qquad U_x(1,y) = y^2.$$

4. Solve the Laplace equation on a disk of radius r_* with boundary condition given by

$$U(r_*, \theta) = \sin^2\theta.$$

HINT: remember that $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$.

5. Use Poisson's formula to compute in closed form the solution to the Laplace equation on the disk of radius r_* with boundary condition

$$U(r_*, \theta) = \begin{cases} U_1 & 0 < \theta < \pi \\ U_2 & \pi < \theta < 2\pi \end{cases}$$

with U_1 and U_2 two constants. In class we obtained the solution in the form of an infinite series.
HINT: divide the Poisson integral in two. Also, observe that for $a > b \geq 0$ one has

$$\int \frac{dx}{a^2 + b^2 - 2ab\cos x} = \frac{2}{a^2 - b^2} \arctan\left(\frac{a+b}{a-b}\tan\frac{x}{2}\right).$$

CHALLENGE: how one would compute the above integral from scratch?

6. Solve the Laplace equation on the annular region

$$\Omega = \{r_1 < r < r_2\}$$

with boundary conditions given by

$$U(r_1, \theta) = C_1,$$
$$U(r_2, \theta) = C_2.$$

What happens when $C_1 = C_2$? Can you give an interpretation of the solution in terms of the temperature of a metallic annular object?

7. Let (r, θ) denote the usual polar coordinates. Show that if $U(r, \theta)$ is a harmonic function (i.e. a solution to the Laplace equation) then also $U(1/r, \theta)$ is a harmonic function.

8. Solve the Laplace equation on the annular region

$$\Omega = \{1 < r < e\}$$

with boundary conditions given by

$$U(1, \theta) = 1 + e \sin \theta,$$
$$U(e, \theta) = 3 + \sin \theta.$$

Here e is the base of the natural logarithms.