PROBLEM SET 6 FOR MTH 6151

1. Solve the initial value problem on the real line

$$\begin{cases} U_{tt} - U_{xx} = \sin x \\ U(x,0) = 0 \\ U_t(x,0) = \sin x \end{cases}$$

2. Solve the initial value problem on the real line

$$\begin{cases} U_{tt} - c^2 U_{xx} = -x \\ U(x,0) = e^{-x^2} \\ U_t(x,0) = 1 \end{cases}$$

3. Solve the following initial value problem on the real line

$$\begin{cases} U_{tt} - c^2 U_{xx} = 1 \\ U(x,0) = 0, U_t(x,0) = 0 \end{cases}$$

4. Solve the following boundary value and initial value problem for wave equation

$$\begin{cases} U_{tt} - c^2 U_{xx} = 0, 0 < x < \pi, t > 0 \\ U(0,t) = 0, U(\pi,t) = 0 \\ U(x,0) = b \sin x + 2b \sin(2x), U_t(x,0) = \sin x \end{cases}$$

5. Solve the following boundary value and initial value problem for wave equation

$$\begin{cases} U_{tt} - c^2 U_{xx} = 0, -\pi < x < \pi, t > 0 \\ U(0,t) = 0, U(\pi,t) = 0 \\ U(x,0) = |\sin x|, U_t(x,0) = 0. \end{cases}$$

6. Consider the wave equation

$$U_{tt} - c^2 U_{xx} = 0$$

on the half-line $x \in [0, \infty)$ with boundary condition

$$U_x(0,t) = 0$$

and initial conditions

$$U(x,0) = f(x), \qquad U_t(x,0) = g(x).$$

In this problem we construct the solution to the above problem using **even extensions**. For this, let

$$F(x) \equiv \begin{cases} f(x) & x \geq 0 \\ f(-x) & x < 0 \end{cases},$$

and

$$G(x) \equiv \begin{cases} g(x) & x \geq 0 \\ g(-x) & x < 0 \end{cases} .$$

(i) Show that $F(x)$ and $G(x)$ are even functions. Show that $F'(x)$ is an odd function.

(ii) Use D'Alembert's formula to write down the solution to the problem

$$V_{tt} - c^2 V_{xx} = 0, \qquad x \in \mathbb{R}$$
$$V(x,0) = F(x), \qquad V_t(x,0) = G(x).$$

(iii) Compute $V_x(x,t)$.

(iv) Show that $V_x(0,t) = 0$. HINT: use that F and G are even and that F' is odd. What can you conclude about $V(x,t)$ in relation to the original problem for $U(x,t)$ on the half line?

(v) Provide an interpretation of what is happening with the string and how reflection works in this case.

7. Proceeding as in the lectures, show that the energy of the solution $U(x,t)$ to the problem

$$U_{tt} - c^2 U_{xx} = 0, \qquad x \geq 0, \qquad t \geq 0,$$
$$U(0,t) = 0,$$
$$U(x,0) = f(x), \qquad U_t(x,0) = g(x),$$

for the wave equation on the half-line is conserved. Assume that $f(x)$, $g(x)$ are compactly supported and $U(x,t)$ is compactly supported for every t (i.e. vanish for sufficiently large x). What happens if the boundary condition is replaced by $U(0,t) = h(t)$ with $h(t) \neq 0$? In this latter case can you provide an interpretation of what is happening in terms of a vibrating string?

8. Consider the following hyperbolic equation on the real line

$$U_{tt} - a^2 U_{xx} - cU_t = 0, c < 0$$

- Assume that $U(x,t)$ is compactly supported for every t. Prove that the the energy of this equation is non-increasing in time.
  (Hint: the energy as is defined in Week 4 lecture notes is $E[U](t) = \frac{1}{2}\int_{-\infty}^{\infty}[U_t^2 + \alpha^2 U_x^2]dx$.)
- Assume also that $\psi(x), f(x), g(x)$ are also compactly supported. Use the fact about energy non-increasing to show the uniqueness of the solution to the inhomogeneous equation on the real line

$$U_{tt} - a^2 U_{xx} - cU_t = \psi(x), c < 0,$$
$$U(x,0) = f(x), \qquad U_t(x,0) = g(x),$$