PROBLEM SET 5 FOR MTH 6151

1. Consider

$$U_{tt} - c^2 U_{xx} = 0$$

for $x \in [0, L]$ with boundary conditions

$$U_x(0,t) = 0, \qquad U(L,t) = 0.$$

Verify the eigenfunctions are

$$X_n(x) = \cos\left(\left(\frac{1}{2} + n\right)\frac{\pi x}{L}\right),$$

and write down the series expansion for a solution $U(x,t)$.

2. Solve

$$U_{tt} - c^2 U_{xx} = 0$$

for $x \in [0, \pi]$ with the boundary conditions

$$U_x(0,t) = U_x(\pi,t) = 0$$

and the initial conditions

$$U(x,0) = 0, \qquad U_t(x,0) = \cos^2 x.$$

Hint:

- $\cos^2 x = \frac{\cos(2x)+1}{2}$.
- Also notice the boundary conditions for this question, compared to the one $U(0,t) = U(L,t) = 0$ in the lecture notes, and the one $U_x(0,t) = U(L,t) = 0$ from Question 1.

3. Find the Fourier series of $f(x) = |x|$ on $[-L, L]$. Draw a sketch of $f(x)$.

4. Find the Fourier series of $f(x) = |\sin x|$ on the interval $[-\pi, \pi]$. Draw a sketch of $f(x)$.

5. Proceeding as in the lectures and using the conservation of energy to show that the solution to the problem

$$\begin{aligned}
&U_{tt} - c^2 U_{xx} = 0, \qquad x \geq 0, \qquad t \geq 0,\\
&U(0,t) = 0,\\
&U(x,0) = f(x), \qquad U_t(x,0) = g(x),
\end{aligned}$$

is unique.

6. Consider for $x \in [0, L]$ the wave equation

$$U_{tt} - c^2 U_{xx} = 0,$$

with boundary conditions

$$U(0, t) = 0, \qquad U_x(L, t) = 0,$$

and the initial conditions

$$U(x, 0) = x, \qquad U_t(x, 0) = 0.$$

Find, using the method of separation of variables, the solution explicitly in series form. HINT: look at Question 1 in this problem set.