PROBLEM SET 4 FOR MTH 6151

1. Use D'Alembert's formula to find the solution to

$$U_{tt} - c^2 U_{xx} = 0$$

with the initial data

$$U(x,0) = e^x,$$
$$U_t(x,0) = \sin x.$$

2. Let $U(x,t)$ denote a solution to the wave equation

$$U_{tt} - c^2 U_{xx} = 0.$$

Show that

$$V(x,t) \equiv U_x(x,t) = \frac{\partial}{\partial x} U(x,t)$$

is also a solution to the wave equation.

3. Let $U(x,t)$ denote a solution to the wave equation

$$U_{tt} - c^2 U_{xx} = 0.$$

Show that

$$V(x,t) \equiv U(x+k, t+h),$$

where k and h are constants is also a solution to the wave equation.

4. Let $U(x,t)$ denote a solution to the wave equation

$$U_{tt} - c^2 U_{xx} = 0.$$

Show that

$$V(x,t) \equiv U(\alpha x, \alpha t)$$

is also a solution to the wave equation for any constant α.

5. Solve

$$U_{xx} - 3U_{xt} - 4U_{tt} = 0,$$
$$U(x,0) = x^2,$$
$$U_t(x,0) = e^x.$$

Hint: factor the operator as it was done in the lectures for the wave equation. This will lead to two consecutive first order pde's with constant coefficients. Each of these equations can be solved using the method of the characteristics.

6. Use the conservation of the energy of the wave equation to show that the only solution to the problem

$$
\begin{aligned}
&U_{tt} - c^2 U_{xx} = 0, \qquad x \in \mathbb{R}, \qquad t \geq 0\\
&U(x,0) = 0,\\
&U_t(x,0) = 0,
\end{aligned}
$$

is given by

$$U(x,t) = 0.$$

Can you give an interpretation of the above result in terms of a vibrating string?

7. Show by direct computation that D'Alembert's formula

$$U(x,t) = \frac{1}{2}\Big(f(x+ct) + f(x-ct) \Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

solves the wave equation

$$U_{tt} - c^2 U_{xx} = 0.$$

HINT: recall that

$$\frac{d}{dz}\int_0^z h(s)ds = h(z).$$

Split in two the integral in D'Alembert's formula.

8. Use D'Alembert's solution to compute the solution to the problem

$$
\begin{aligned}
&U_{tt} - c^2 U_{xx} = 0,\\
&U(x,0) = 0,\\
&U_t(x,0) = \frac{1}{1+x^2}.
\end{aligned}
$$

Provide a sketch of the solution for different times. What do you think happens with the string?

9. Solve the Goursat problem:

$$
\begin{cases}
U_{tt} - c^2 U_{xx} = 0\\
u|_{x-ct=0} = x^2\\
u|_{x+ct=0} = x^4.
\end{cases}
$$

Hint: Use the formula for general solutions of wave equation on the real line.