Week 2.

Example X and Y are r.v.'s with joint pdf given by

$$f_{X,Y}(s,t) = \begin{cases} e^{-s-t}, & \text{if } s,t \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find

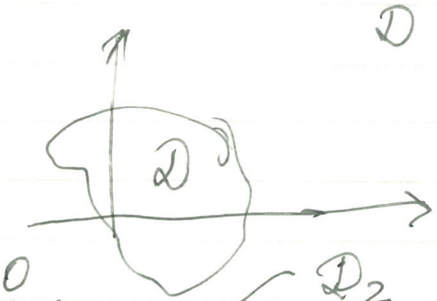
(i) $\mathbb{P}(3 < X < 4 \text{ and } 2 < Y < 5) = p_1$

(ii) $\mathbb{P}(-1 < X < 3 \text{ an } 2 < Y < 5) = p_2$

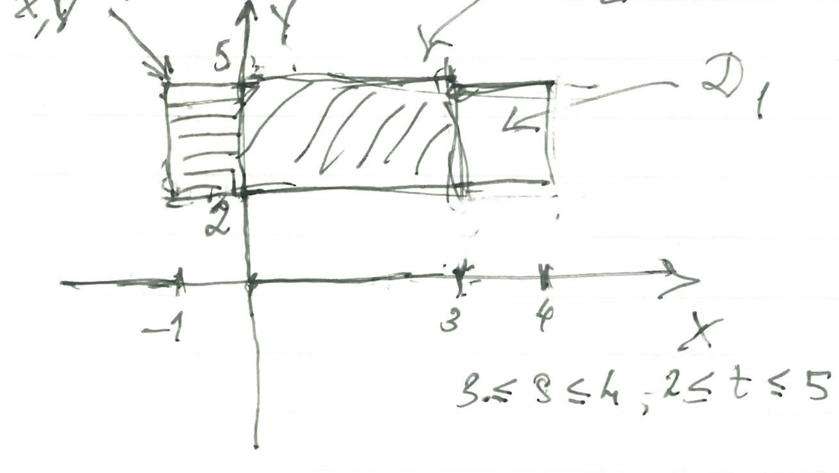
(iii) $\mathbb{P}(Y > X > 0) = p_3$

Solution. By definition of the pdf,

$$\mathbb{P}((X,Y) \in D) = \iint_D f_{X,Y}(s,t)\,ds\,dt$$



(i)



$$P_1 = \int_3^4 \int_2^5 e^{-s-t}\, ds\, dt =$$

$$\int_3^4 e^{-s}\left(\int_2^5 e^{-t}\, dt\right) ds =$$

$$\int_3^4 e^{-s}\left(-e^{-t}\Big|_{t=2}^{5}\right) ds =$$

$$\left(-e^{-t}\Big|_{t=2}^{5}\right) = e^{-2} - e^{-5}$$

$$(e^{-2} - e^{-5}) \int_3^4 e^{-s}\, ds = (e^{-2} - e^{-5})(e^{-3} - e^{-4})$$

$$= 0.04$$

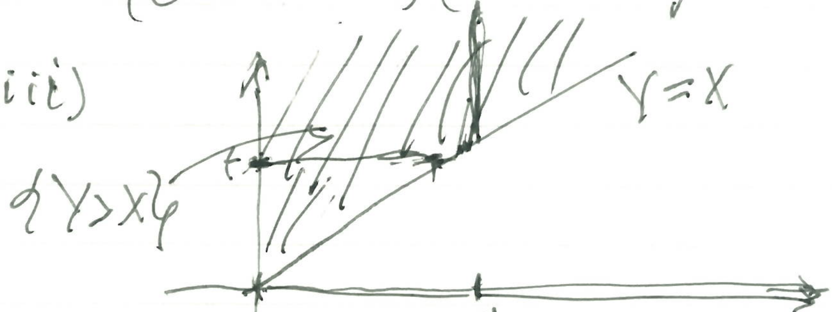
(ii) $P_2 = P(0 \le X < 3 \text{ and } 2 < Y < 5) =$

$$\int_0^3 \left(\int_2^5 e^{-s-t}\, dt\right) ds =$$

$$\int_0^3 e^{-s}(e^{-2} - e^{-5})\, ds =$$

$$(e^{-2} - e^{-5})(1 - e^{-3}) = 0.122$$

(iii)



1) $0 < s < \infty \quad s < t < \infty$

2) $0 < t < \infty \quad 0 < s < t$

$$P_3 = \mathbb{P}(Y > X) = \int_0^{\infty}\left(\int_s^{\infty} e^{-s-t}\,dt\right)ds = \int_0^{\infty}\left(\int_0^{t} e^{-s-t}\,ds\right)dt$$

So

$$P_3 = \int_0^{\infty} e^{-s}\Bigg(\underbrace{\int_s^{\infty} e^{-t}\,dt}_{\displaystyle -e^{-t}\Big|_s^{\infty} = -e^{-\infty}-(-e^{-s}) = e^{-s}}\Bigg)ds$$

Hence

$$P_3 = \int_0^{\infty} e^{-2s}\,ds = -\frac{1}{2}e^{-2s}\Big|_0^{\infty} = \frac{1}{2}.$$

<u>Remark.</u>

$$P_2 = \int_{-1}^{3}\left(\int_2^5 f_{X,Y}(s,t)\,dt\right)ds = \underbrace{\int_{-1}^{0}\int_2^5}_{0} + \int_0^3\left(\int_2^5 e^{-s-t}\,dt\right)ds$$

Cumulative distribution functions

I. The one-dimensional case

<u>Def-n.</u> The cumulative distribution function of a r.v. X is given by

$$F_X(x) \overset{def}{=} \mathbb{P}(X \le x)$$

If X is a cont. r. v. with pdf $f_X(t)$, then

$$\boxed{F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt} \qquad (1)$$

If we know $F_X(x)$, then, differentiating (1) over x gives

$$\boxed{F_X'(x) = f_X(x)} \qquad \boxed{f_X(x) = \frac{dF_X(x)}{dx}}$$

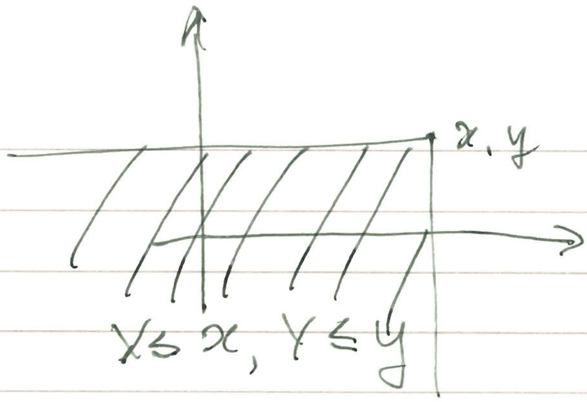
II The two-dimensional case.

Def-n. The joint cumulative distribution function of two r.v.'s X, Y is given by

$$F_{X,Y}(x,y) \overset{def}{=} \mathbb{P}(X \le x, Y \le y)$$

Corollary. If $f_{X,Y}(s,t)$ is the joint pdf of (X, Y), then

$$\boxed{F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)\,ds\,dt} \qquad (2)$$



<u>Theorem</u> Suppose that X, Y are jointly continuous r.v.'s with cdf $F_{X,Y}(x)$. Then the pdf of X, Y is given by

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

<u>Proof</u>. We differentiate (a). Then

$$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} F_{X,Y}(x,y)\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} \int_{-\infty}^{x} \left(\int_{-\infty}^{y} f_{X,Y}(s,t)\,dt\right) ds\right)$$

Let $\varphi(s) = \int_{-\infty}^{y} f_{X,Y}(s,t)\,dt$. Then

$$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} \int_{-\infty}^{x} \varphi(s)\,ds\right) =$$

$$\frac{\partial}{\partial y}\varphi(x) = \frac{\partial}{\partial y}\int_{-\infty}^{y} f_{X,Y}(x,t)\,dt = f_{X,Y}(x,y)$$

∎

Marginal distributions.

If we know $f_{X,Y}(x,y)$, we would like to be able to find $f_X(x)$, $f_Y(y)$.

The distribution of X or Y is called the marginal distribution.

Remark. $F_X(x) = F_{X,Y}(x, +\infty)$ because the events $\{X \le x\} = \{X \le x, Y < \infty\}$ are the same. (because Y is always $< \infty$).

Theorem. If $f_{X,Y}(x,y)$ is the pdf of X, Y then

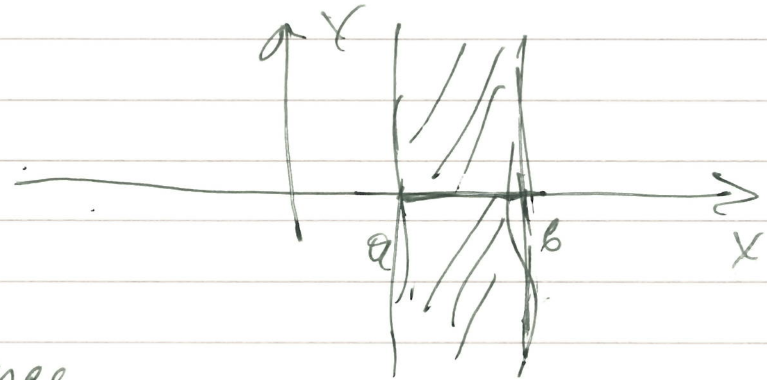
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \qquad (3)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \qquad (4)$$

Proof. Recall that $f_X(x)$ is a funct of x s.t.

$$\mathbb{P}(X \in [a,b]) = \int_a^b f_X(x)\,dx$$

But $\mathbb{P}(X \in [a,b]) = \mathbb{P}(X \in [a,b], Y \in (-\infty,\infty))$



Hence

$$\mathbb{P}(X \in [a,b]) = \int_a^b \underbrace{\left(\int_{-\infty}^{\infty} f_{X,Y}(s,t)\,dt\right)}_{\varphi(s)} ds = \int_a^b \varphi(s)\,ds$$

Hence $\varphi(s) = f_X(s)$ or

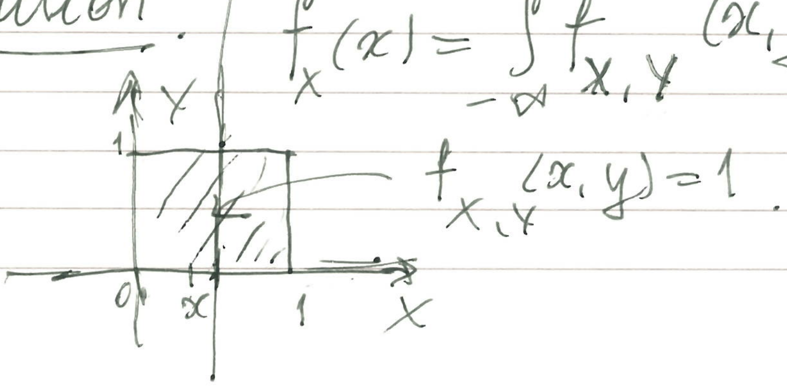
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy. \qquad \blacksquare$$

<u>Exercise</u>. Prove (4).

<u>Example</u>. $f_{X,Y}(x,y) = \begin{cases} 1 & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise} \end{cases}$

Find $f_X(x)$ and $f_Y(y)$.

<u>Solution</u>. $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$



1) If $x \leq 0$, then $f_{X,Y}(x,y) = 0 \Rightarrow$

$$f_X(x) = \int_{-\infty}^{\infty} 0 \, dy = 0$$

2) If $x \geq 1$, then $f_{X,Y}(x,y) = 0 \Rightarrow$

$$f_X(x) = 0.$$

3) If $0 < x < 1$ then

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 f_{X,Y}(x,y)\,dy$$

$$= \int_0^1 1\,dy = y\Big|_{y=0}^{1} = 1.$$

Hence

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

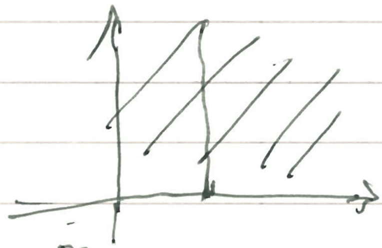
Similarly

$$f_Y(y) = \begin{cases} 1 & \text{if } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

<u>Exercise</u> Compute $f_X(y)$.

Example $f_{X,Y}(x,y) = \begin{cases} e^{-x-y} & \text{if } x>0,\ y>0 \\ 0 & \text{otherwise.} \end{cases}$

Find $f_X(x)$ and $f_X(y)$.



Solution.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \begin{cases} \int_0^{\infty} e^{-x-y}\,dy & \text{if } x>0 \\ 0 & \text{if } x \le 0 \end{cases}$$

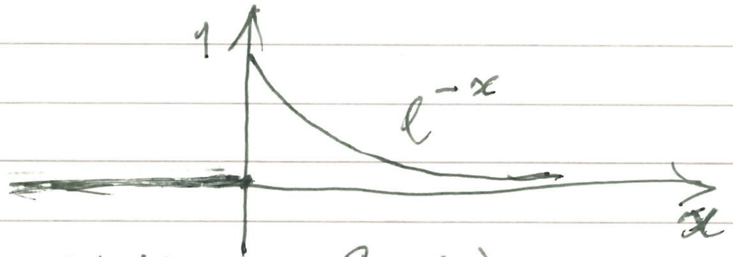
$$\int_0^{\infty} e^{-x-y}\,dy = e^{-x}\int_0^{\infty} e^{-y}\,dy = e^{-x}\left(-e^{-y}\Big|_{y=0}^{\infty}\right) = e^{-x}$$

Answer: $f_X(x) = \begin{cases} e^{-x} & \text{if } x>0 \\ 0 & \text{if } x \le 0 \end{cases}$

Similarly,

$$f_Y(y) = \begin{cases} \int_0^{\infty} e^{-x-y}\,dx & \text{if } y>0 \\ 0 & \text{if } y \le 0 \end{cases}$$

$$= \begin{cases} e^{-y} & \text{if } y>0 \\ 0 & \text{if } y \le 0 \end{cases}$$



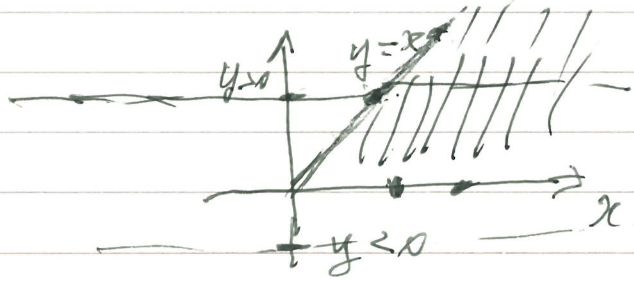
X, Y are $Exp(1)$

$X \sim Exp(\lambda)$

$f_X(x) = e^{-\lambda x},\ x>0$

<u>Example</u>. $f_{X,Y}(x,y) = \begin{cases} 2e^{-x-y} & \text{if } x \geq y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Find $f_Y(y)$



Solution

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

If $y < 0$, then $f_{X,Y}(x,y) = 0$. Hence

$$f_Y(y) = \int_{-\infty}^{\infty} 0\,dx = 0.$$

If $y \geq 0$, then

$$f_Y(y) = \underbrace{\int_{-\infty}^{y} 0\,dx}_{0} + \int_{y}^{\infty} 2e^{-x-y}\,dx$$

$$= 2e^{-y} \int_{y}^{\infty} e^{-x}\,dx$$

$$= 2e^{-y} \left(-e^{-x} \Big|_{x=y}^{\infty} \right)$$

$$= 2e^{-y} \left(-e^{-\infty} - (-e^{-y}) \right)$$

$$= 2e^{-y} \times e^{-y} = 2e^{-2y}.$$

<u>Answer</u>: $f_Y(y) = \begin{cases} 2e^{-2y}, & \text{if } y \geq 0 \\ 0, & \text{if } y < 0 \end{cases}$

$Y \sim Exp(2)$

<u>Exercise</u>. Find $f_X(x)$.

<u>Expectations over joint distributions</u>

We are given $f_{X,Y}(x,y)$ for X, Y.
($f_{X,Y}$ is the pdf of r.v.'s X, Y).
$Z = g(X,Y)$ is a new r.v.
How do we find $\mathbb{E}(Z)$?
Here $g : \mathbb{R}^2 \to \mathbb{R}$.

<u>Theorem</u>. Suppose that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g(x,y)| f_{X,Y}(x,y)\,dx\,dy < \infty$

Then the expection of $Z = g(X,Y)$ is given by

(5) $$\mathbb{E}(Z) = \mathbb{E}(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

<u>Example</u>.

<u>Theorem</u>. $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$.

<u>Proof</u>. By (5)

$$\mathbb{E}(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_{X,Y}(x,y)\,dx\,dy$$

$$= \iint x f_{X,Y}(x,y)\,dx\,dy + \iint y f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} x \underbrace{\left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\right)}_{=f_X(x)} dx + \int_{-\infty}^{\infty} y \underbrace{\left(\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\right)}_{f_Y(y)} dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx + \int_{-\infty}^{\infty} y f_Y(y)\,dy = \mathbb{E}(X) + \mathbb{E}(Y)$$

☒

Remark. In our case $g(x,y) = x+y$.

Covariance and Correlation.

Example. $g(x,y) = (x-\mu_1)^k (y-\mu_2)^m$,

where $\mu_1 = \mathbb{E}(X)$, $\mu_2 = \mathbb{E}(Y)$,

and $f_{X,Y}(x,y)$ is the pdf of X, Y.

How do we find $\mathbb{E}(X-\mu_1)^k(Y-\mu_2)^m$?

Answer. If $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(x-\mu_1)^k(y-\mu_2)^m| f_{X,Y}(x,y)\,dx\,dy$ ($< \infty$)

(is $< \infty$) then, by (5)

$$\mathbb{E}\left[(X-\mu_1)^k(Y-\mu_2)^m\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_1)^k(y-\mu_2)^m f_{X,Y}(x,y)\,dx\,dy$$

↑ mixed central moment of (X, Y)

Special case: $k=1$, $m=1$. Then we obtain the covariance of X, Y:

$$\mathrm{Cov}(X,Y) \stackrel{\text{def}}{=} \mathbb{E}\left[(X-\mu_1)(Y-\mu_2)\right]$$

<u>Theorem</u>

$$\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mu_1\mu_2.$$

<u>Exercise</u>. Prove this theorem.

Exercise class

a) Bern (p) X – the number of successes in one experiment. $X = 0$ or 1.

$$\mathbb{P}(X=1) = p, \quad \mathbb{P}(X=0) = 1-p = q$$

$$\mathbb{E}(X) = 1 \times p + 0 \times q = p. \quad Var(X) = pq$$

b) Binomial (n, p)

X	0	1	...	n
Prob	q^n			p^n

$$\mathbb{P}(X=1) = \binom{n}{1} p^1 q^{n-1} = npq^{n-1}$$

$$\mathbb{P}(X=k) = \binom{n}{k} p^k q^{n-k}$$

$$\mathbb{E}(X) = np, \quad Var(X) = npq$$

$$X_1, X_2, \dots, X_n$$

$$X = X_1 + X_2 + \dots + X_n.$$

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \dots + \mathbb{E}(X_n) = np.$$

$$Var(X) = Var(X_1) + \dots + Var(X_n) = npq.$$

$$Var(X) = \mathbb{E}(X-\mu)^2 = \mathbb{E}(X^2) - \mu^2, \quad \mu = \mathbb{E}(X)$$

If $X \sim Bern(p)$, then

$$Var(X) = \mathbb{E}(X^2) - p^2 = \mathbb{E}(X) - p^2 = p - p^2 = p(1-p) = pq$$

$$X^2 = X$$

(c) $\mathbb{P}(X=k) = pq^{k-1}, \quad k=1,2,\dots$

$$E(X) = 1\cdot p + 2pq + 3pq^2 + \dots kpq^{k-1} + \dots$$

$$= p\left(\frac{d}{dq}\left(q + q^2 + q^3 + \dots q^k + \dots\right)\right)$$

$$= p\frac{d}{dq}\left(\frac{q}{1-q}\right) = p\frac{1}{(1-q)^2} = \frac{1}{p}.$$

d) Poisson(λ)

$$\mathbb{P}(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad \lambda > 0.$$

$$k = 0, 1, \dots.$$

e) Gamma (α, β), $X \sim Ga(\alpha, \beta)$

$$f_X(x) = \begin{cases} \beta^\alpha x^{\alpha-1} e^{-\beta x}, & x > 0 \\ 0, & x \le 0. \end{cases}$$

$$\beta > 0, \quad \alpha > 0$$

$$\int_0^\infty \beta^\alpha x^{\alpha-1} e^{-\beta x}\, dx = \int_0^\infty \beta^\alpha \left(\frac{y}{\beta}\right)^{\alpha-1} e^{-y} \frac{dy}{\beta}$$

$$\beta x = y \quad dx = \frac{dy}{\beta}, \quad x = \frac{y}{\beta}$$

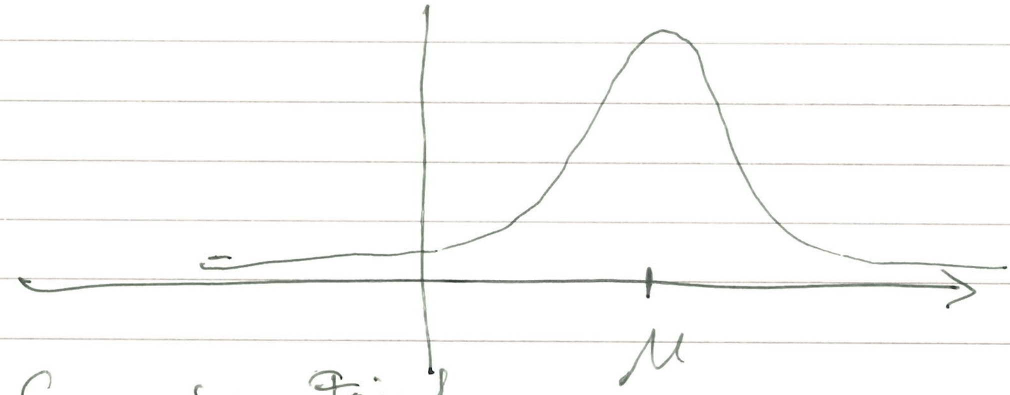
$$0 < y < \infty$$

$$= \int_0^\infty y^{\alpha-1} e^{-y}\, dy = \Gamma(\alpha) \Rightarrow$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1.$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = 1 \iff$$

$$\boxed{\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \sqrt{2\pi}\,\sigma}$$



Exercise. Find

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{a^2}}\, dx \stackrel{?}{=}$$

$a > 0$