Note that the point -2 is an eventually fixed point, since $f(-2) = 4 - 2 = 2$, and 2 is a fixed point.

Note also that 0 is an eventually fixed point, since $f(0) = -2$

$$\therefore f^2(0) = f(-2) = 2$$

We can represent this as

$$\begin{tikzcd} 0 \arrow[r, "f"] & -2 \arrow[r, "f"] & 2 \arrow[loop right, "f"] \end{tikzcd}$$

Similarly, 1 is an eventually fixed point (i.e. an eventually periodic pt of period 1), as are $\sqrt{3}$ and $-\sqrt{3}$.

$$\begin{tikzcd} \sqrt{3} \arrow[dr, "f"] & & \\ & 1 \arrow[r, "f"] & -1 \arrow[loop right, "f"] \\ -\sqrt{3} \arrow[ur, "f"'] & & \end{tikzcd}$$

Que Are there points of (least) period 2? If so, what are they?

Ans Points x of period 2 satisfy

$$f^2(x) = x$$

ie. $f(f(x)) = x$

ie. $f(x^2-2) = x$

ie. $(x^2-2)^2 - 2 = x$

ie. $x^4 - 4x^2 + 4 - 2 = x$

ie. $x^4 - 4x^2 - x + 2 = 0 \quad (*)$

Note that

$(x+1)(x-2) = x^2 - x - 2$

is a factor of the LHS of $(*)$

In fact we can factorise the LHS of $(*)$ as

$$(x^2 - x - 2)(x^2 + x - 1)$$

This means that there are <u>two</u> points of <u>least</u> period 2, and they are precisely the roots of the quadratic

$$x^2 + x - 1,$$

i.e. $$x = \frac{1}{2}\left(-1 \pm \sqrt{1^2 - 4(-1)}\right)$$

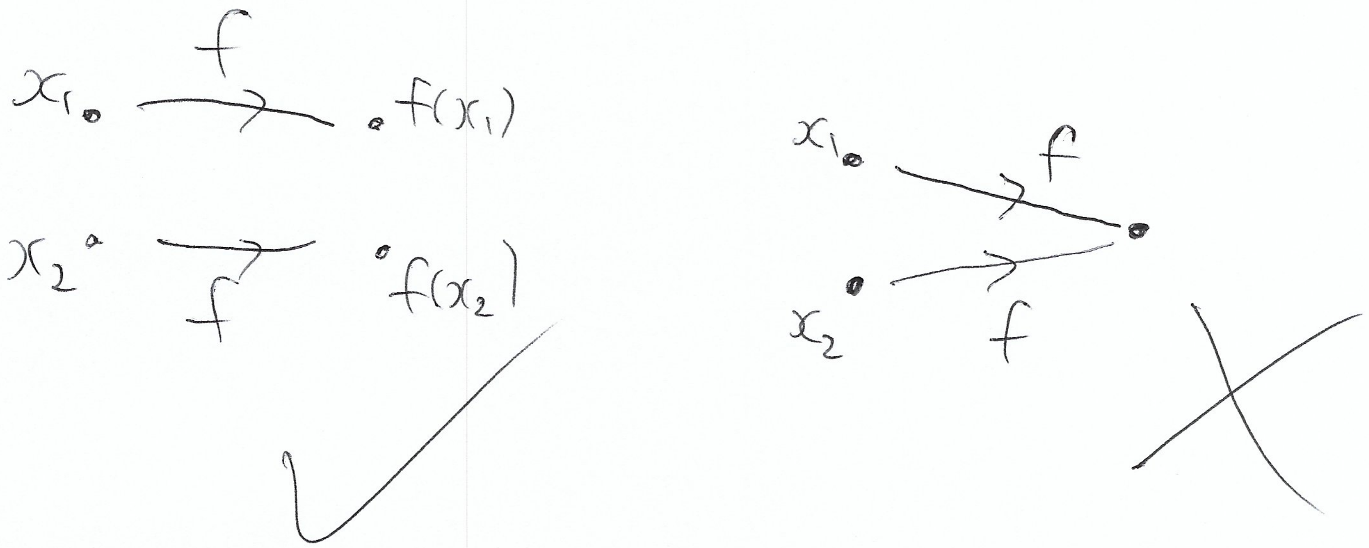
$$= \tfrac{1}{2}\left(-1 \pm \sqrt{5}\right)$$

So we have a 2-cycle

$$\left\{ \tfrac{1}{2}(-1+\sqrt{5}),\ \tfrac{1}{2}(-1-\sqrt{5}) \right\}.$$

<u>Recall</u> Let A, B be sets. $\left(\begin{array}{c}\text{e.g.}\\ A=B=\mathbb{R}\end{array}\right)$

A function $f: A \to B$ is said to be <u>one-to-one</u> if, for all $x_1, x_2 \in A$ with $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.



A function $f: A \to B$ is <u>onto</u> if for all $y \in B$, there exists $x \in A$, such that $f(x) = y$.

A function $f: A \to B$ is <u>invertible</u> (or <u>bijective</u>) if it is both one-to-one and onto. In this case there is an <u>inverse</u> function $f^{-1}: B \to A$, i.e. such that $f^{-1}(f(x)) = x$ for all $x \in A$

and $f(f^{-1}(y)) = y$ " $y \in B$.

Lemma If $f : \mathbb{R} \to \mathbb{R}$ is one-to-one then every eventually periodic point is actually a periodic point.

Remark Among our examples so far, $f(x) = x^2$ and $f(x) = x^2 - 2$ are not one-to-one, but $f(x) = -x^3$ is one-to-one.

Proof of Lemma: Let x_0 be eventually periodic of period n.

So there exists some $k \geq 0$ such that $x_k = f^k(x_0)$ is a period-n pt.

$$\text{i.e. } f^n\big(f^k(x_0)\big) = f^k(x_0)$$

$$\text{i.e. } f^k\big(f^n(x_0)\big) = f^k(x_0) \quad (*)$$

~~Inject~~ The fact that f is one-to-one means that $f : \mathbb{R} \to f(\mathbb{R})$ is invertible, with $\underbrace{f(\mathbb{R})}_{:= \{f(x) : x \in \mathbb{R}\}}$

inverse map $f^{-1} : f(\mathbb{R}) \to \mathbb{R}$

i.e. satisfying $f^{-1}(f(x)) = x \quad \forall x \in \mathbb{R}$,

and hence $(f^{-1})^k (f^k(x)) = x \quad \forall x \in \mathbb{R}$

If we apply $(f^{-1})^k$ to $(*)$, we get

$$(f^{-1})^k(f^k(f^n(x_0))) = (f^{-1})^k(f^k(x_0))$$

$$\text{i.e. } f^n(x_0) = x_0 .$$

In other words, x_0 is a period-n point.

So we have shown that the eventually periodic point x_0 is in fact periodic, as required.

$\square$

Corollary If $f: \mathbb{R} \to \mathbb{R}$ is invertible (i.e. bijective) then every eventually periodic point is actually a periodic point.

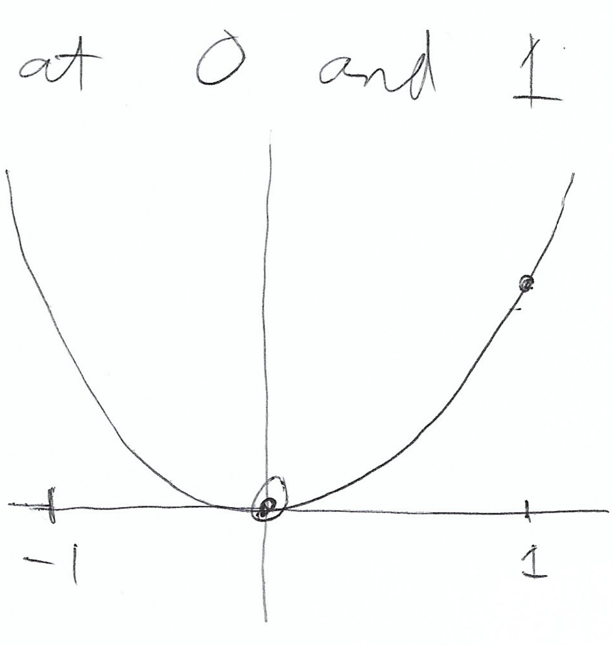
Definition Let p be a fixed point of f. The basin of attraction of p is defined to be the set of points

$$\mathrm{Basin}(p) := \{x_0 \in \mathbb{R} : \lim_{n\to\infty} f^n(x_0) = p\}$$

Example $f(x) = x^2$. Recall this f has fixed points at 0 and 1.

Here,

$$\mathrm{Basin}(0) = (-1, 1)$$
$$= \{x_0 : -1 < x_0 < 1\}$$



Qu What is Basin(1)?

Ans Basin(1) = $\{-1, 1\}$

We can extend the definition of basin of attraction to periodic orbits:

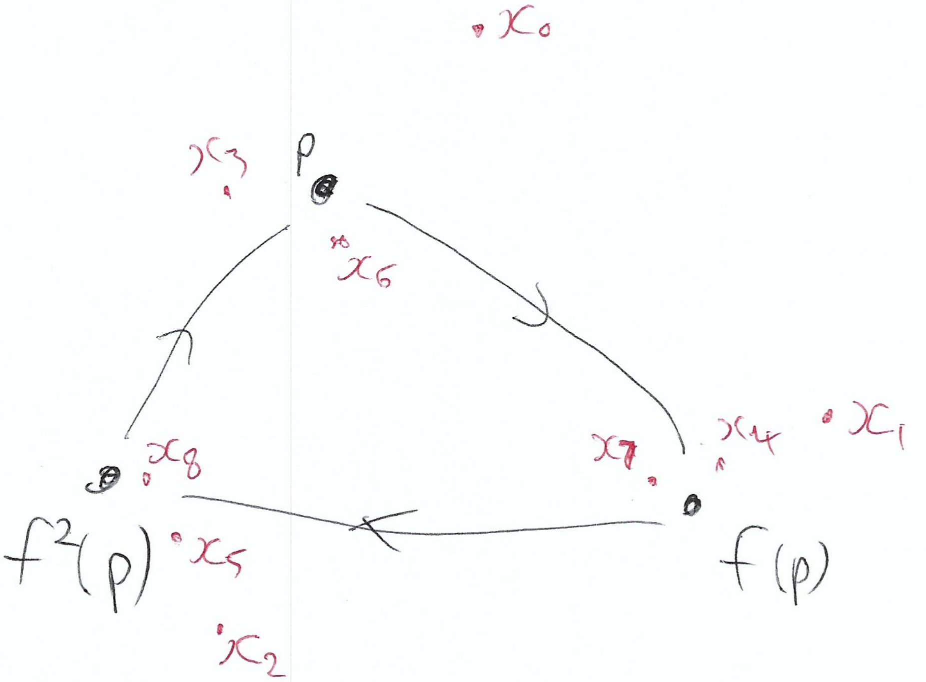
Defn Let p be a periodic point of least period k under the map f. The basin of attraction of the k-cycle $\{p, f(p), \ldots, f^{k-1}(p)\}$ is the set

$$\text{Basin}(\{p, f(p), \ldots, f^{k-1}(p)\})$$

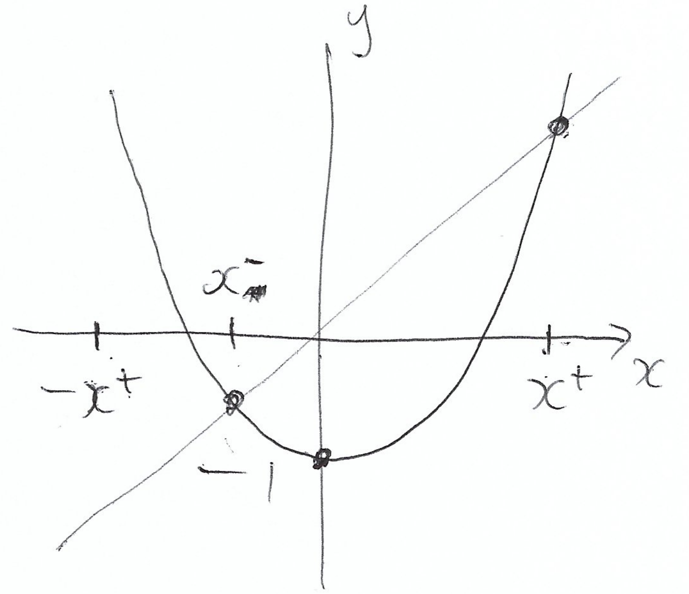
$$:= \bigcup_{i=0}^{k-1} \{x_0 \in \mathbb{R} : \lim_{n \to \infty} f^{nk}(x_0) = f^i(p)\}$$

Schematic diagram : (illustrating a point x_0 in the basin of attraction of a periodic orbit)

$(k=3)$



Example Consider the map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 1$.



Fixed points:

$$f(x) = x$$

i.e. $x^2 - 1 = x$

i.e. $x^2 - x - 1 = 0$

i.e. $x = \frac{1}{2}\left(1 \pm \sqrt{1+4}\right)$

$$= \frac{1}{2}\left(1 \pm \sqrt{5}\right)$$

Let us write $x^+ = \frac{1}{2}\left(1 + \sqrt{5}\right) \simeq 1.62$

and $x^- = \frac{1}{2}\left(1 - \sqrt{5}\right) \simeq -0.62$

these are the 2 fixed points of f.

Claim: Points in the union

$$(-\infty, -x^+) \cup (x^+, \infty)$$

have orbits which converge to ∞

(i.e. if $x_0 \in (-\infty, -x^+) \cup (x^+, \infty)$

then $f^n(x_0) \to \infty$ as $n \to \infty$)

Question: What happens if $x_0 \in (-x^+, x^+)$?

It turns out that there is a 2-cycle

$\{0, -1\}$.

The fixed point $x^+ = \frac{1+\sqrt{5}}{2}$ is "repelling".

The fixed point $x^- = \frac{1-\sqrt{5}}{2}$ is also "repelling".

Question: If $x_0 \in (-1, 0)$, what happens to $f^n(x_0) = x_n$?

$x_0 = -0.5$

$x_1 = (-0.5)^2 - 1 \quad = -0.75$

$x_2 = (-0.75)^2 - 1 \quad = -0.4375$

$x_3 = \quad -0.8085\ldots$

$x_4 = \quad -0.345\ldots$

$x_5 = \quad -0.8801\ldots$

$x_6 = \quad -0.225\ldots$

$x_7 = \quad -0.949\ldots$

$x_8 = \quad -0.098\ldots$

$x_9 = \quad -0.9902\ldots$

$x_{10} = \quad -0.0194\ldots$

We 'see' that $x_0 = -1/2$ belongs to the basin of attraction of $\{-1, 0\}$

In fact, ~~the~~ $\text{Basin}(\{-1, 0\})$ is equal to $(-x^+, x^+) \setminus \{x^-\}$

$$= (-x^+, x^-) \cup (x^-, x^+)$$

Attracting and repelling fixed points

<u>Defn</u> Let p be a fixed point of a function $f : \mathbb{R} \to \mathbb{R}$. We say that p is <u>attracting</u> if there exists $\delta > 0$ such that $I = (p - \delta, p + \delta) \subset \text{Basin}(p)$.

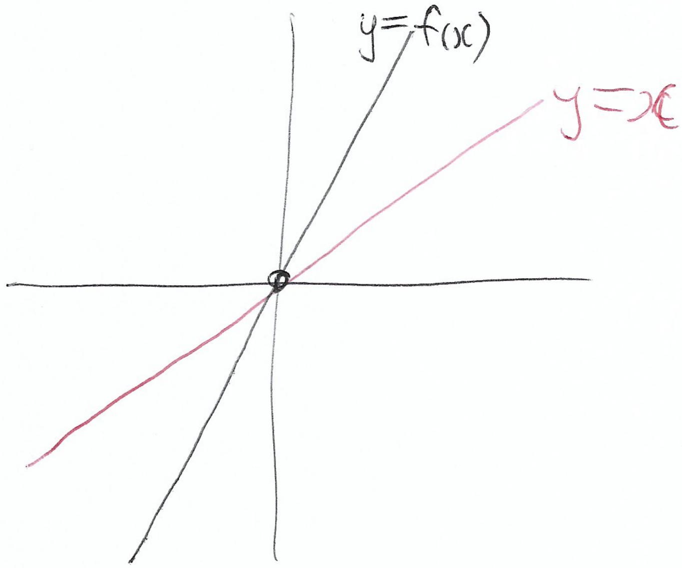
In other words, p is attracting if $\exists \delta > 0$ s.t. for all $x \in (p - \delta, p + \delta)$,

$$\lim_{n \to \infty} f^n(x) = p.$$

Defn The fixed point p is repelling if there exists $\delta > 0$ such that if $I = (p-\delta, p+\delta)$ then for all $x \in I \setminus \{p\}$ there exists $N \in \mathbb{N}$ such that $f^N(x) \notin I$.

In other words, p is repelling if there exists an open interval I centred on p for which every point in $I \setminus \{p\}$ escapes from that interval (under iteration by f).

Example Let $f(x) = ax$, where $a \in \mathbb{R}$.



Note that 0 is a fixed point of f.

Note that $f^2(x) = f(f(x)) = f(ax) = a^2 x$,

and in general $f^n(x) = a^n x$.

- Suppose $|a| < 1$.

  In this case 0 is an attracting fixed point. To see this, choose any $\delta > 0$, let $x_0 \in I = (-\delta, \delta)$.

  Then $\lim_{n\to\infty} f^n(x_0) = \lim_{n\to\infty} a^n x_0 = 0$,

  since $|a| < 1$

- Suppose $|a| > 1$.

In this case 0 is <u>repelling</u>.

To see this, let $\delta > 0$, and let $x_0 \in (-\delta, \delta) \setminus \{0\}$.

Then
$$\begin{aligned} |f^n(x_0)| &= |a^n x_0| \\ &= |a|^n |x_0| \\ &> \delta \end{aligned}$$
for all n sufficiently large.

<u>Example</u> Let $f(x) = x^2$.

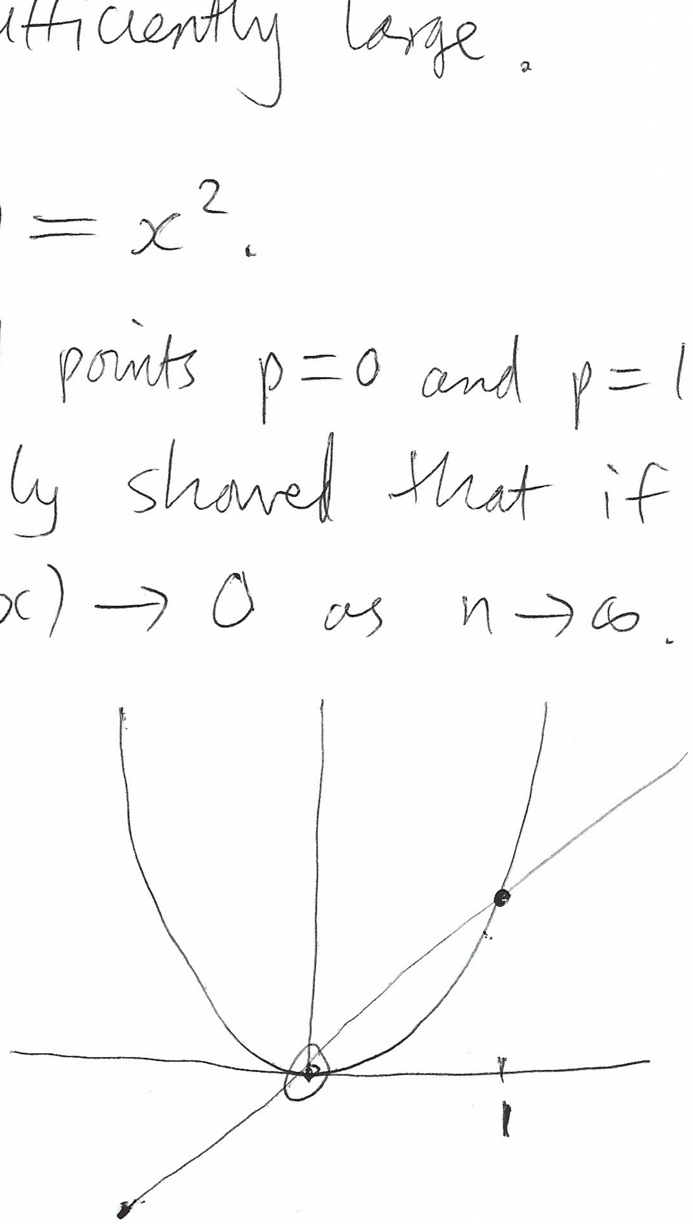
Recall this has fixed points $p=0$ and $p=1$.

For $p=0$ we previously showed that if $|x| < 1$ then $f^n(x) \to 0$ as $n \to \infty$.

In other words, such a point x lies in Basin(0).

i.e. $(-1,1) \subset$ Basin(0)

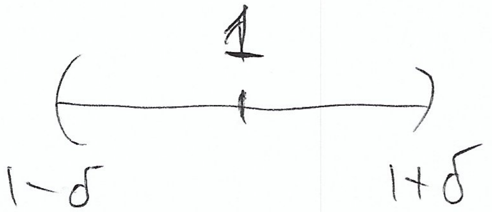
Thus 0 is <u>attracting</u>.



For $p = 1$, we used 'cobweb diagram analysis' to guess it is a repelling fixed point.

To make this rigorous, we could choose some $\delta > 0$ (e.g. $\delta = 1/3$) and show that if $x_0 \in (1-\delta, 1+\delta) \setminus \{1\}$

$$= (1-\delta, 1) \cup (1, 1+\delta)$$



then there exists $N \in \mathbb{N}$ such that $f^N(x_0) \notin (1-\delta, 1+\delta)$

Definition A function $f: \mathbb{R} \to \mathbb{R}$ is said to be C^1 if f is differentiable, and its derivative f' is continuous.

Theorem If $f: \mathbb{R} \to \mathbb{R}$ is C^1, and p is a fixed point of f, then

- p is attracting if $|f'(p)| < 1$
- p is repelling if $|f'(p)| > 1$

Note This theorem is inspired by the example of $f(x) = ax$, where we saw that 0 is attracting if $|a| < 1$, and 0 is repelling if $|a| > 1$.

(In this case $f'(x) = a$, so in particular $f'(0) = a$)