Selected Solutions to Problem Set 2.

1. (1). We use the method of change of variables

$$\begin{cases} \tilde{x} = x - 2t \\ \tilde{t} = -2x - t \end{cases}$$

then $u_x = \frac{\partial u}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial x} + \frac{\partial u}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial x} = u_{\tilde{x}} - 2u_{\tilde{t}}$

$u_t = \frac{\partial u}{\partial \tilde{x}} \frac{\partial \tilde{x}}{\partial t} + \frac{\partial u}{\partial \tilde{t}} \frac{\partial \tilde{t}}{\partial t} = -2u_{\tilde{x}} - u_{\tilde{t}}$

The equation then becomes

$$(u_{\tilde{x}} - 2u_{\tilde{t}}) - 2(-2u_{\tilde{x}} - u_{\tilde{t}}) = 0$$

Namely $5u_{\tilde{x}} = 0$

Integrate ~~$u_{\tilde{x}} = f$~~ and get

$$\boxed{u = f(\tilde{t}) = f(-2x-t)}$$

2. (1). It's the same equation as in 1(1), we ~~use~~ use the method of characteristic here.

the characteristic lines are,

$$\frac{dt}{dx} = \frac{-2}{1}$$

so they are stright lines

$$\boxed{t = -2x + C} \quad \text{or} \quad \boxed{2x + t = C}$$

The solution only depends on ~~the~~ which characteristic line it's on, so

$$u(x,t) = \hat{f}(C) = \hat{f}(2x+t) = f(-2x-t)$$

where $\boxed{\hat{f}(s) = f(-s).}$

Now we specify f by the ~~initial~~ boundary condition.
when $x=0$, $t=c$, and so

$$\cos c = \cos t = u(0,t) = f(-t) = f(-c)$$

$$\text{so } f(c) = \cos(-c) \text{ and } f(-c) = \cos c$$

and the ~~general~~ solution is

$$\underline{u(x,t)} = f(-2x-t) = f(-c) = \cos c$$

$$= \underline{\cos(2x+t)}$$

4. The change of variable we use is

$$\begin{cases} \tilde{x} = ax+by \\ \tilde{y} = bx-ay. \end{cases}$$

so

$$\begin{cases} u_x = a u_{\tilde{x}} + b u_{\tilde{y}} \\ u_y = b u_{\tilde{x}} - a u_{\tilde{y}} \end{cases}$$

The Left hand side of equation becomes

$$\cancel{a\,u_x +}\ a(a u_{\tilde{x}} + b u_{\tilde{y}}) + b(b u_{\tilde{x}} - a u_{\tilde{y}}) + cu = 0$$

Namely $(a^2+b^2)u_{\tilde{x}} + cu = 0$

$$u_{\tilde{x}} + \frac{c}{a^2+b^2}u = 0.$$

Using integrating factor $\left(e^{\frac{c}{a^2+b^2}\tilde{x}}\right)$, we have

$$\frac{\partial}{\partial \tilde{x}}\left(e^{\frac{c}{a^2+b^2}\tilde{x}}u\right) = 0$$

$$e^{\frac{c}{a^2+b^2}\tilde{x}}u = f(\tilde{y})$$

$$u = f(\tilde{y}) \cdot e^{-\frac{c}{a^2+b^2}\tilde{x}}$$

$$= f(bx-ay) \cdot e^{-\frac{c}{a^2+b^2}(ax+by)}$$

is the general solution

6. Do change of variable

$$\begin{cases} \tilde{x} = x+y \\ \tilde{y} = x-y \end{cases}$$

we get $\begin{cases} x = \frac{1}{2}(\tilde{x}+\tilde{y}) \\ y = \frac{1}{2}(\tilde{x}-\tilde{y}) \end{cases}$

LHS of the equation becomes

$$u_x + u_y + u = \left(u_{\tilde{x}} \cdot \frac{\partial \tilde{x}}{\partial x} + u_{\tilde{y}} \frac{\partial \tilde{y}}{\partial x}\right) + \left(u_{\tilde{x}} \frac{\partial \tilde{x}}{\partial y} + u_{\tilde{y}} \frac{\partial \tilde{y}}{\partial y}\right) + u$$

$$= \cancel{2u_{\tilde{x}} + 2u}$$

$$\cancel{u_{\tilde{x}}}\ u_{\tilde{x}} + u_{\tilde{y}} + u_{\tilde{x}} - u_{\tilde{y}} + u$$

$$= 2u_{\tilde{x}} + u$$

RHS of the equation is

$$e^{x+2y} = e^{\frac{1}{2}(\tilde{x}+\tilde{y}) + (\tilde{x}-\tilde{y})} = e^{\frac{3\tilde{x}}{2} - \frac{\tilde{y}}{2}}$$

The equation becomes

$$2u_{\tilde{x}} + u = e^{\frac{3\tilde{x}}{2} - \frac{\tilde{y}}{2}}$$

Integrate ~~we get~~ Using the integrating factor $e^{\frac{x^2}{2}}$, we get

$$\frac{\partial}{\partial x}\left(e^{\frac{x^2}{2}}u\right) = \frac{1}{2} e^{-\frac{1}{2}y^2} \cdot e^{2x^2}$$

$$e^{\frac{x^2}{2}} \cdot u = \frac{1}{2} \int e^{2x^2} \cdot e^{-\frac{1}{2}y^2} dx$$

$$u = \frac{1}{2} e^{-\frac{x^2}{2}} \cdot e^{-\frac{y^2}{2}} \cdot \left[\frac{1}{2} e^{2x^2} + f(y^2)\right]$$

$$u(x,y) = \frac{1}{4} e^{\frac{3x^2}{2} - \frac{y^2}{2}} + \frac{1}{2} e^{-\frac{x^2}{2} - \frac{y^2}{2}} \cdot f(y^2)$$

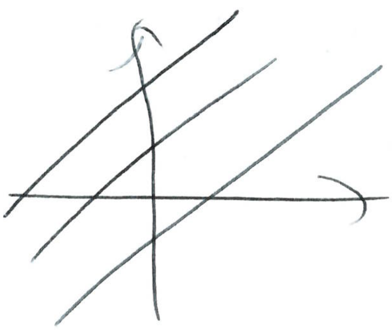
$$= \frac{1}{4} e^{x+2y} + \frac{1}{2} e^{-\frac{x+y}{2}} f(x-y)$$

8. The characteristic satisfy the ODE

$$\frac{dy}{dx} = 1.$$

the charateristic curves are then

$$y = x + C$$



The equation becomes

$$u_x + u_y$$

$$= u_x + \frac{y}{x} u_y$$

$$= u_x + \frac{\partial y}{\partial x} u_y$$

$$= \frac{d}{dx} u$$

$$= 1$$

so $u(x,y) = x + f(c)$

$= x + f(y-x).$