

Main Examination period 2023 – May/June – Semester B

## MTH6105 / MTH6105P: Algorithmic Graph Theory

Duration: 2 hours

The exam is intended to be completed within **2 hours**. However, you will have a period of **4 hours** to complete the exam and submit your solutions.

**You should attempt ALL questions. Marks available are shown next to the questions.**

All work should be **handwritten** and should **include your student number**. Only one attempt is allowed – **once you have submitted your work, it is final**.

In completing this assessment:

- You may use books and notes.
- You may use calculators and computers, but you must show your working for any calculations you do.
- You may use the Internet as a resource, but not to ask for the solution to an exam question or to copy any solution you find.
- You must not seek or obtain help from anyone else.

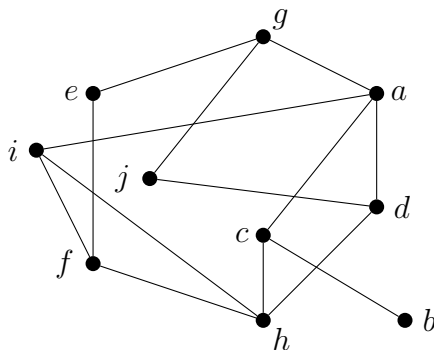
When you have finished:

- scan your work, convert it to a **single PDF file**, and submit this file using the tool below the link to the exam;
- e-mail a copy to **maths@qmul.ac.uk** with your student number and the module code in the subject line;

Examiners: F. Fischer, V. Latora

You may use any result from lecture notes and exercises without proving it, but you must state clearly which result you use.

**Question 1 [24 marks].** Let  $G$  be the graph given by the following drawing.



- (a) Draw the induced subgraph of  $G$  on vertex set  $\{a, d, e, f, g, h\}$ . [4]
- (b) Draw a subgraph of  $G$  that is isomorphic to the graph  $H$  with  $V(H) = \{r, s, t, u, v, w, x, y, z\}$  and  $E(H) = \{rs, rt, ru, st, sv, tw, ux, vy, wz\}$ . [4]
- (c) Draw a spanning tree of  $G$  whose set of leaves is  $\{a, b, g, j\}$ , or explain why such a spanning tree does not exist. [4]

Call a cycle of a graph  $H$  a **Hamiltonian cycle** of  $H$  if it contains every vertex of  $H$ .

- (d) Give a Hamiltonian cycle of  $G$ , or explain why such a cycle does not exist. [4]

Let  $G$  be an arbitrary simple graph,  $n = |V(G)|$ , and  $m = |E(G)|$ .

- (e) Assume that the complement of  $G$  is connected. Show that  $m \leq \frac{1}{2}n^2 - \frac{3}{2}n + 1$ . [8]

**Question 2 [24 marks].** Let  $D$  be a directed graph.

(a) Assume that  $D$  is a directed acyclic graph. Prove or disprove that there exists a vertex  $v \in V(D)$  such that  $d_D^+(v) = 0$ . [4]

(b) Assume that there exists a vertex  $v \in V(D)$  such that  $d_D^+(v) = 0$ . Prove or disprove that  $D$  must be a directed acyclic graph. [4]

Consider the directed network  $(D, w)$  with  $V(D) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ ,  
 $A(D) = \{v_1v_3, v_1v_5, v_2v_4, v_2v_6, v_3v_2, v_3v_6, v_4v_6, v_5v_2, v_5v_3\}$ , and

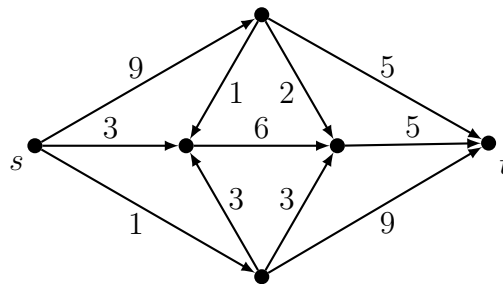
$$\begin{aligned} w(v_1v_3) &= 5, & w(v_1v_5) &= 4, & w(v_2v_4) &= 1, & w(v_2v_6) &= 3, & w(v_3v_2) &= 2, \\ w(v_3v_6) &= 5, & w(v_4v_6) &= 1, & w(v_5v_2) &= 7, & w(v_5v_3) &= 3. \end{aligned}$$

(c) Draw the network. [4]

(d) Show that  $D$  is a directed acyclic graph. [4]

(e) Use Morávek's algorithm to find a longest  $v_1-v_6$ -path of  $(D, w)$ . Show your working and give the path and its length. [8]

**Question 3 [26 marks].** Consider the directed network  $(D, c)$  given by the following drawing, where each arc  $e \in A(D)$  is labeled by its capacity  $c(e)$  and two vertices  $s$  and  $t$  have been identified.



- (a) Use the Ford-Fulkerson algorithm to find a maximum  $s-t$ -flow of  $(D, c)$ . Draw the residual network after each iteration of the algorithm, and give the size of the maximum flow. [10]
- (b) Use a cut to show that the flow you have found is indeed a maximum  $s-t$ -flow of  $(D, c)$ . [6]
- (c) If the capacity of both of the arcs with capacity 9 was decreased to 6, how would this affect the size of a maximum flow? Justify your answer. [4]

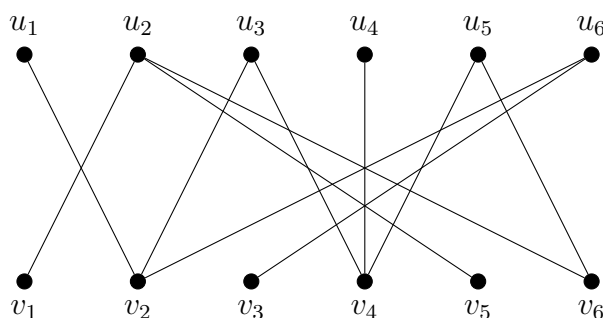
Call a directed network  $(D, c)$  an  $s-t$ -**bottleneck network**, where  $s, t \in V(D)$ , if for every arc  $uv \in A(D)$  there exists a minimum  $s-t$ -cut  $S$  of  $(D, c)$  with  $u \in S$  and  $v \notin S$ .

- (d) Give an efficient algorithm that determines, for a given directed network  $(D, c)$  with  $c : A(D) \rightarrow \mathbb{N}$  and  $s, t \in V(D)$ , whether  $(D, c)$  is an  $s-t$ -bottleneck network. Explain briefly why the algorithm is correct and efficient. [6]

**Question 4 [26 marks].**

- (a) Give a tree  $T$  with  $|V(T)| = 7$  that has exactly two maximum matchings. Justify your answer. [4]
- (b) Let  $G$  be an acyclic graph with  $V(G) \neq \emptyset$ . Show that there exists  $v \in V(G)$  with  $d_G(v) = 0$ , or there exist  $u, v \in V(G)$  with  $u \neq v$  and  $d_G(u) = d_G(v) = 1$ . [4]
- (c) Let  $T$  be a tree,  $M_1$  and  $M_2$  distinct maximum matchings of  $T$ . Show that  $|M_1| + |M_2| < |V(T)|$ . [6]

Consider the tree  $T$  given by the following drawing.



- (d) Find a maximum matching of  $T$ . Show your working. [8]
- (e) Give a set  $X \subseteq \{u_1, u_2, u_3, u_4, u_5, u_6\}$  with  $|X| > |N_T(X)|$ , or explain why such a set does not exist. [4]

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**End of Paper.**