Queen Mary
University of London

MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 2 (Real Numbers)

1. Consider the following sets:

   (a) $A = [-1, 3]$.
   (b) $B = (-1, 3)$.
   (c) $C = (-1, 3) \cap [-3, 1]$.
   (d) $D = (1, 2) \cup [7, 8]$.
   (e) $E = \{z \in \mathbb{R} : z^3 < 2\}$.
   (f) $F = \{n^2 : n \in \mathbb{N}\}$.
   (g) $G = \{z \in \mathbb{R} : 0 < z^2 < 1\}$.

   For each of (a)-(g), answer the following questions (fully justify your answers):

   (i) Does this set have an upper bound?
   (ii) Does this set have a supremum?
   (iii) Does this set have a maximum?
   (iv) Does this set have a lower bound?
   (v) Does this set have an infimum?
   (vi) Does this set have a minimum?

2. Let $A = \{1/n : n \in \mathbb{N}\}$.

   (a) Find, with brief justification, a lower bound for A.
   (b) Suppose $x \in \mathbb{R}$ with $x > 0$. Is x a lower bound for A? Justify your answer. (You may use any theorems from the course providing you clearly state which theorem you are using.)
   (c) Does A have an infimum? Prove your answer.

3. Suppose $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$ are sets, and that $a = \sup A$ and $b = \sup B$ both exist.

   (a) Prove that $A \cap B$ is bounded above by a and also by b. (This means that a and b are both upper bounds for $A \cap B$.)

(b) Suppose $A \cap B \neq \emptyset$. Prove that $A \cap B$ has a supremum, m say, and that $m \leq \min\{a, b\}$.

(c) Assuming $A \cap B \neq \emptyset$, is it necessarily the case that $m = \min\{a, b\}$? Either give a proof or give a counterexample.

Something to think about (not part of the question). What happens in part (b) if $A \cap B = \emptyset$?

4. First, restudy our proof of Theorem 2.19 from the lecture notes (stating that there exists a number $x \in \mathbb{R}$ with $x^2 = 2$). Modify the proof of Theorem 2.19 to show that there is a real number x with $x^2 = 19$.

5. **Challenge.** Let I_1, I_2, I_3, $\ldots$ be a *decreasing sequence of nested closed intervals*, i.e.,

   - For all $n \in \mathbb{N}$, $I_n = [a_n, b_n]$ is a closed interval.
   - $\forall n \in \mathbb{N} : I_{n+1} \subseteq I_n$.
   - $\forall \varepsilon > 0 \; \exists n \in \mathbb{N} : |I_n| < \varepsilon$, where $|I_n| = b_n - a_n$ is the length of the interval.

   Show, using the Completeness Axiom, that there exists exactly one $x \in \mathbb{R}$ such that $\forall n \in \mathbb{N} \; : \; x \in I_n$ (this is known as the "nested interval principle").

6. **Challenge.** In Question 5, we proved that the completeness axiom implies the nested interval principle. Now, prove that the two are actually *equivalent* by showing that the nested interval principle implies the completeness axiom.