MTH 744 U/P

Coursework 4

Solutions

Dynamical Systems

Coursework 4 - Notes on Solutions

Ex 5.1.2

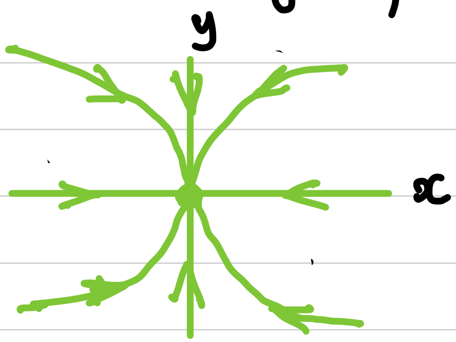
$\dot{x} = ax, \dot{y} = -y$ with $a < -1$

We have $\frac{dx}{dt} = ax, \frac{dy}{dt} = -y$

$\Rightarrow \frac{dy}{-y} = \frac{dx}{ax}$ $(= dt)$. Integrating gives

$\ln(x) + \ln(y^a) = c \Rightarrow x \cdot y^a = e^c = c'$

$\therefore x = c' y^{-a}, a < -1 \Rightarrow x = c' y^b, b > 1$



"orbits/trajectories/solution curves" tangent to y-axis as $t \nearrow$

Ex 5.13

$\dot{x} = -y, \; \dot{y} = -x \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Ex 5.1.5

$\dot{x} = 0, \; \dot{y} = x + y \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

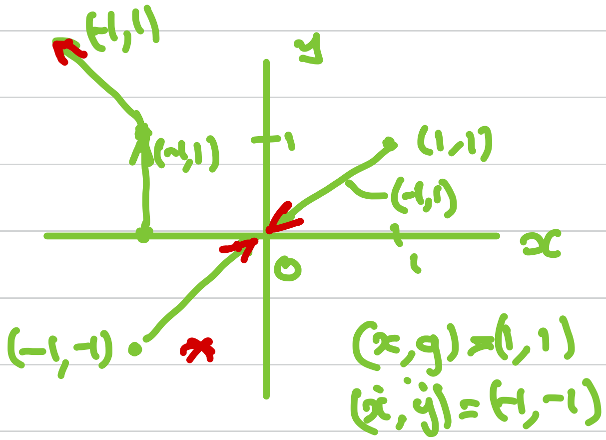
Ex 5.1.7

$\dot{x} = x, \; \dot{y} = x + y \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Ex 5.1.9

$\dot{x} = -y, \; \dot{y} = -x$

$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

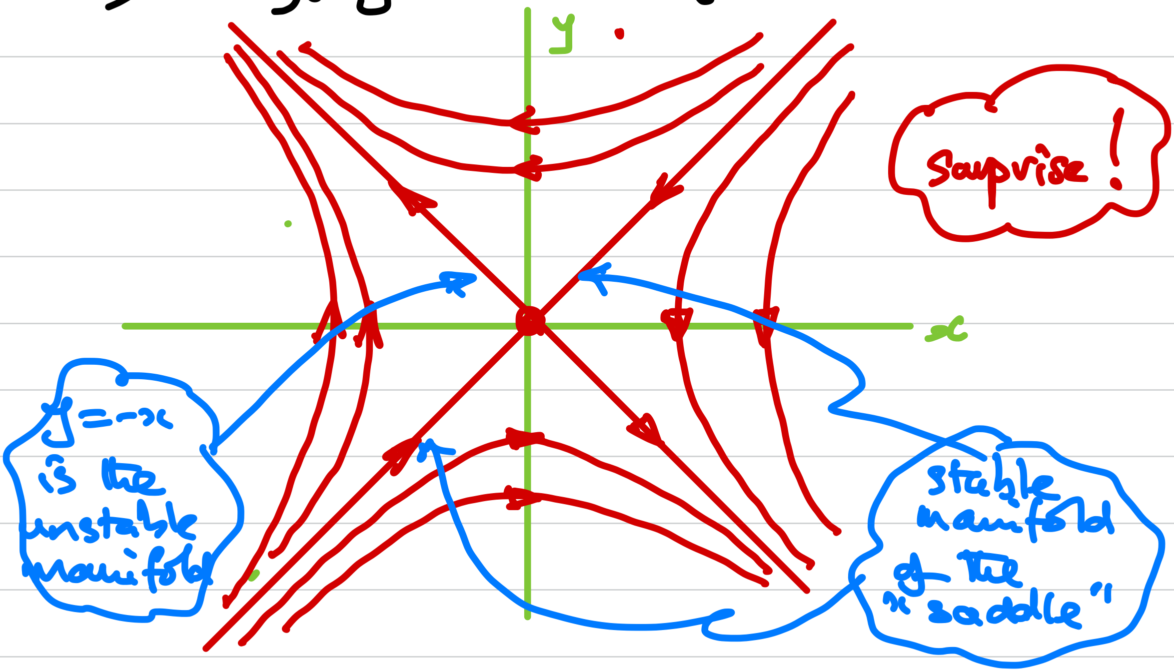


* Note having just a few vectors gives you little idea of the nature of the solution curves try vectorfield plot $(-y, -x)$

To find shape of curves, let's try

$\dot{x} = -y, \quad \dot{y} = -x \Rightarrow \dfrac{dx}{-y} = \dfrac{dy}{-x}$

$\int x\,dx - \int y\,dy = 0 \Rightarrow x^2 - y^2 = c$



Coefficient matrix gives $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, eigenvalues $\begin{vmatrix} \lambda - 0 & +1 \\ +1 & \lambda - 0 \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda_1 = +1, \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = -1, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ex 5.1.10

(a) $\dot{x} = y$, $\dot{y} = -4x$ $\quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

eigenvalues: $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i \Rightarrow$ "centre"

"STABLE, but not asymptotically stable"

(b) $\dot{x} = 0$, $\dot{y} = x$ $\quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

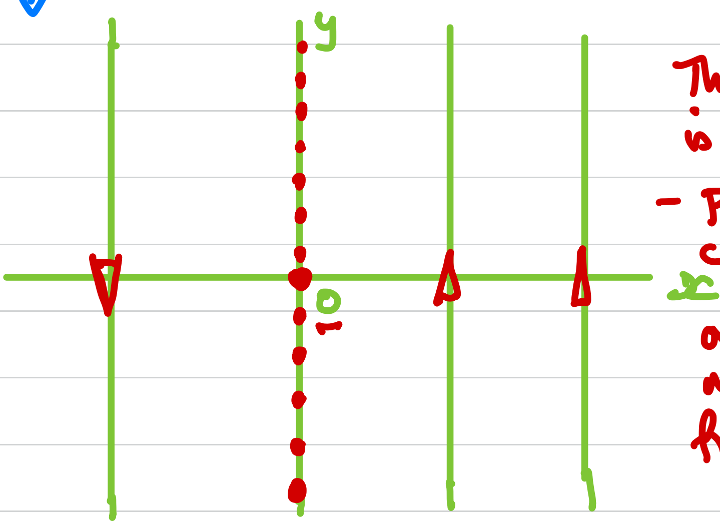
eigenvalues: $\lambda^2 = 0$

eigenvector, just one $(y=0) \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Note line of fixed point on the y-axis.

All trajectories are vertical lines ($x = \text{const}$)

$\dot{y} \gtrless 0$ $\uparrow\downarrow$ for $x \gtrless 0$



The origin is unstable – pts arbitrarily close have orbits which move away from $\underline{0}$

(e) $\dot{x} = -x, \quad \dot{y} = -5y$ $\quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1 & 0^{*} \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

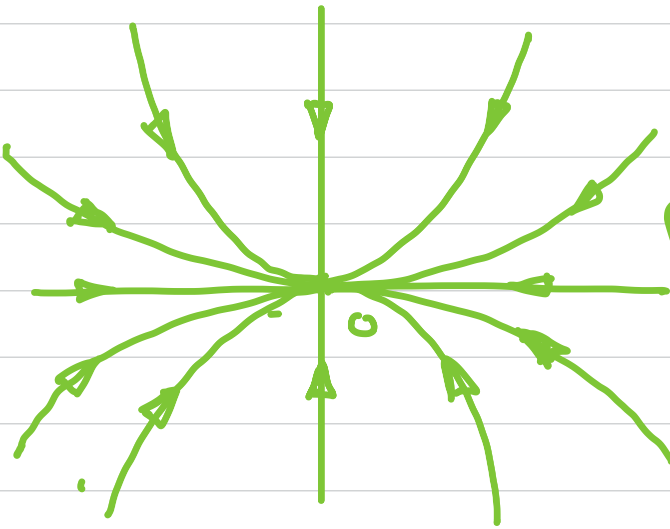
Note $\quad Tr = -6, \quad Det = 5, \quad \lambda = \dfrac{-6 \pm \sqrt{36-20}}{2}$

$\lambda_1 = (-6+4)/2, \quad \lambda_2 = (-6-4)/2$

$\quad = -2/2 \qquad\qquad = -10/2$

$\quad = -1 \qquad\qquad\quad = -5$

$v_1 = (1,0)^T \qquad v_2 = (0,1)^T$



STABLE & ASYMPTOTICALLY STABLE