Selected Solutions to problem set 1

1. (1) 1st order Linear

   (3) 2nd order Linear

   (5) 2nd order Nonlinear

   (7) 2nd order Nonlinear.

2. (1) Linear.

   (2) Nonlinear

   (3) Linear

3. (2) linear and homogeneous.

   (4) Nonlinear and homogeneous

4. u_1, u_2 solves the inhomogeneous linear PDE $\boxed{\mathcal{L}u = g}$ means

$$\mathcal{L}u_1 = g \quad ①$$

$$\mathcal{L}u_2 = g. \quad ②$$

① - ② gives

$$\mathcal{L}u_1 - \mathcal{L}u_2 = g - g = 0$$

by linearity, we have

$$\mathcal{L}v = \mathcal{L}(u_1 - u_2) = \mathcal{L}u_1 - \mathcal{L}u_2 = 0$$

So ~~u_1~~ v solves the homogeneous equation $\mathcal{L}v = 0$.

7. $$u_t = \frac{\partial}{\partial t}\,\text{sech}^2(x-t)$$

$$= 2\,\text{sech}(x-t)\cdot\frac{\partial}{\partial t}\,\text{sech}(x-t)$$

$$= 2\,\text{sech}(x-t)\cdot[-(-\tanh(x-t)\,\text{sech}(x-t))]$$

(chain Rule)

$$= 2\,\text{sech}^2(x-t)\tanh(x-t)$$

similarly

$$u_x = \frac{\partial}{\partial x}\,\text{sech}^2(x-t)$$

$$= 2\,\text{sech}(x-t)\cdot\frac{\partial}{\partial x}\,\text{sech}(x-t)$$

$$= -2\,\text{sech}^2(x-t)\tanh(x-t)$$

2nd derivative

$$u_{xx} = -4\,\text{sech}(x-t)\tanh(x-t)\cdot\frac{\partial}{\partial x}\,\text{sech}(x-t)$$

$$- 2\,\text{sech}^2(x-t)\,\frac{\partial}{\partial x}\tanh(x-t)$$

$$= 4\,\text{sech}^2(x-t)\tanh^2(x-t) - 2\,\text{sech}^2(x-t)[1-\tanh^2(x-t)]$$

$$= 6\,\text{sech}^2(x-t)\tanh^2(x-t) - 2\,\text{sech}^2(x-t)$$

And third derivative

$$u_{xxx} = 12\,\text{sech}(x-t)\tanh^2(x-t)\cdot\frac{\partial}{\partial x}\,\text{sech}(x-t)$$

$$+12\,\text{sech}^2(x-t)\tanh(x-t)\cdot\frac{\partial}{\partial x}\tanh(x-t)$$

$$-4\,\text{sech}(x-t)\,\frac{\partial}{\partial x}\,\text{sech}(x-t)$$

$$= -12\,\mathrm{sech}^2(x-t)\tanh^3(x-t) + 12\,\mathrm{sech}^2(x-t)\tanh(x-t)$$

$$-12\,\mathrm{sech}^2(x-t)\tanh^3(x-t) + 4\,\mathrm{sech}^2(x-t)\tanh(x-t)$$

$$= -24\,\mathrm{sech}^2(x-t)\tanh^3(x-t) + 16\,\mathrm{sech}^2(x-t)\tanh(x-t)$$

Thus. $4u_t + u_{xxx} + 12uu_x$

$$= 8\,\mathrm{sech}^2(x-t)\tanh(x-t) - 24\,\mathrm{sech}^2(x-t)\tanh^3(x-t)$$

$$+16\,\mathrm{sech}^2(x-t)\tanh(x-t) + 12\,\mathrm{sech}^2(x-t)\left[-2\,\mathrm{sech}^2(x-t)\tanh(x-t)\right]$$

$$= 24\,\mathrm{sech}^2(x-t)\tanh(x-t) - 24\,\mathrm{sech}^2(x-t)\tanh^3(x-t)$$

$$-24\,\mathrm{sech}^4(x-t)\tanh(x-t)$$

Using $\mathrm{sech}^2(x-t) = 1 - \tanh^2(x-t)$, we get.

$4u_t + u_{xxx} + 12u \cdot u_x$

$$= 24\,\mathrm{sech}^2(x-t)\tanh(x-t) - 24\,\mathrm{sech}^2(x-t)\tanh^3(x-t)$$

$$-24\,\mathrm{sech}^2(x-t)\tanh(x-t)\cdot\left[1-\tanh^2(x-t)\right]$$

$$= 24\,\mathrm{sech}^2(x-t)\tanh(x-t) - 24\,\mathrm{sech}^2(x-t)\tanh^3(x-t)$$

$$-24\,\mathrm{sech}^2(x-t)\tanh(x-t) + 24\,\mathrm{sech}^2(x-t)\tan^3(x-t)$$

$$= 0\ !$$