Discrete random variables (r.v.'s)

$(\Omega, \mathcal{F}, P)$ – probability space

A r.v. X is a function on Ω

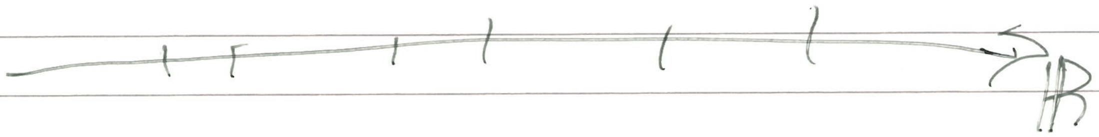
$X : \Omega \to \mathbb{R}$.

Def-n. Discr. r.v.'s are r.v.'s with the following properties.

(1) The values of X belong to a discrete set of numbers.

$P(X = x) > 0$ for each x from this set.

This set can be finite or infinite countable.



$$\sum_x \mathbb{P}(X = x) = 1$$

Expectation and Variance

Def.: $\mathbb{E}(X) \stackrel{\text{def}}{=} \sum_x x \, \mathbb{P}(X=x)$, if $\sum_x |x| \, \mathbb{P}(X=x) < \infty$

Consider a r.v. which is a function of X:

$$Y = g(X), \quad g: \mathbb{R} \to \mathbb{R}.$$

Then, by def-n

$$\mathbb{E}(Y) = \sum_y y \, \mathbb{P}(Y=y), \quad \text{if } \sum_y |y| \, \mathbb{P}(Y=y) < \infty$$

Theorem. If X is discrete r.v., $g: \mathbb{R} \to \mathbb{R}$ is a function, and $\sum_x |g(x)| \, \mathbb{P}(X=x) < \infty$, $Y = g(X)$, then

$$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \sum_x g(x) \, \mathbb{P}(X=x)$$

Exercise. Prove this theorem for the case when X take values $x_1, x_2, \dots, x_k$.

Remark. The $\mathbb{E}(X) = x_1 \mathbb{P}(X=x_1) + \dots + x_k \mathbb{P}(X=x_k)$

$$\left(= \sum_{x_j} x_j \, \mathbb{P}(X=x_j)\right)$$

Remark. This Th-m is important because, in particular, it allows one to compute $\mathbb{E}(Y)$, $Y = g(X)$, without computing $\mathbb{P}(Y = y)$.

Continuous R.V.'s

For cont. r.v., $\mathbb{P}(X = x) = 0$ for all x.

We consider the probabilities

$$\mathbb{P}(X \in [a, b])$$

Def-n. A r.v. X is continuous if there is a function $f_X(x)$ which has two properties:

(i) $f_X(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

and

$$\mathbb{P}(X \in [a, b]) = \int_a^b f_X(x)\,dx.$$

for any interval $[a, b]$.

$f_X(x)$ is the probability density function (pdf)

Expectation.

Def-n. If X is a cont. r.v. with pdf (probability density function) $f_X(x)$ then

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx \quad \text{if} \int_{-\infty}^{\infty} |x| f_X(x) < \infty$$

If $g: \mathbb{R} \to \mathbb{R}$, we can define

$$Y = g(X) \text{ - a new r.v.}$$

Theorem. In this setting

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx,$$

$$\text{if} \int_{-\infty}^{\infty} |g(x)| f_X(x)\,dx < \infty.$$

Remark. $\mathbb{E}(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy$

This Theorem allows one to compute $\mathbb{E}(Y)$ without computing $f_Y(y)$ first.

Conditional Probability (CP)

1. Conditional probability of events

Def-n. Suppose that A and B are events. Given that $P(B) > 0$

$$P(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Def-n. Events $B_1, B_2, \dots, B_n$ form a partion if

(i) $B_i \cap B_j = \emptyset$ for all $i \neq j$

(ii) $\bigcup_{j=1}^{n} B_j = \Omega$

(Recall: events are subsets of Ω).

If ω is an **elementary** event, $\omega \in \Omega$, then it belongs to exactly one of the B_j's.

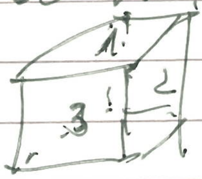
Theorem. (The Total Probability Th-m)

Suppose that A is an event and $B_1, B_2, \dots, B_n$ form a partition. Then

$$P(A) = \sum_{j=1}^{n} P(B_j) P(A|B_j).$$

Example. You roll two dice.

What is the pr-ty that the product is even?



$1, 2, \ldots, 6$

$B_1, B_2, \ldots, B_6$ – form a partition.

B_j = the even of obtaining j at 1st roll.

$$\mathbb{P}(B_j) = \frac{1}{6}.$$

A – the event that the product is even.

$$\mathbb{P}(A|B_1) = \frac{3}{6} = \frac{1}{2}. = \mathbb{P}(A|B_3) = \mathbb{P}(A|B_5)$$

$$\mathbb{P}(A|B_2) = 1. = \mathbb{P}(A|B_4) = \mathbb{P}(A|B_6)$$

Hence

$$\mathbb{P}(A) = \frac{1}{6}\sum_{j=1}^{6}\mathbb{P}(A|B_j) = \frac{1}{6}\left(3\times 1 + 3\times\frac{1}{2}\right)$$

$$= \frac{1}{6}(4.5) = \frac{4.5}{6} = \frac{9}{12} = \frac{3}{4}.$$

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another approach:

B_1 – the first result is odd.

B_2 —— " —— is even.

(see notes).

Example. Alice and Bob play a game. They roll a fair dice. If it comes up 1 or 2, Alice wins. If it comes up 3, Bob wins. If it comes up 4, 5, or 6 they continue rolling until one of them wins.

What is the pr-ty that Alice wins?

Solution. Consider events

$B_1 = \{1, 2\}$, $B_2 \{3\}$, $B_3 = \{4, 5, 6\}$, (at first roll).

B_1, B_2, B_3 form a partition.

A = event that Alice wins.

$\mathbb{P}(B_1) = \frac{2}{6} = \frac{1}{3}$, $\mathbb{P}(B_2) = \frac{1}{6}$, $\mathbb{P}(B_3) = \frac{1}{2}$.

$\mathbb{P}(A|B_1) = 1$, $\mathbb{P}(A|B_2) = 0$,

$\mathbb{P}(A|B_3) = \mathbb{P}(A)$ (!). Hence

$$\mathbb{P}(A) = \frac{1}{3} \times 1 + \frac{1}{6} \times 0 + \frac{1}{2}\mathbb{P}(A) \Rightarrow$$

$$\frac{1}{2}\mathbb{P}(A) = \frac{1}{3} \Rightarrow \boxed{\mathbb{P}(A) = \frac{2}{3}}.$$

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Conditional Distribution of Discrete R.V.'s.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Let X be a discrete r.v.

$P(X=x)$ is given

Question. What can we say about the distribution of X, given that B occurs?

Answer. We can compute

$$P(X=x|B) = \frac{P((X=x) \cap B)}{P(B)}$$

Definition $Y = X|B$ is a new r.v. whose distribution is given by $P(Y=x) = P(X=x|B)$

Example We toss a fair coin 2 times. $X = \#$ of heads. What is the conditional distribution of X given that the 1^{st} toss is tails?... is heads?

Answer. $B_1 = \{ 1^{st} \text{ toss is heads} \}$

$B_2 = \{ 1^{st} \text{ toss is tails} \}$

(i) $Y_1 = X | B_1$, $X = 0, 1, 2$ - possible values of X.

$$\mathbb{P}(Y_1 = 0) = \mathbb{P}(X = 0 | B_1) = 0$$

$$\mathbb{P}(Y_1 = 1) = \mathbb{P}(X = 1 | B_1) = \frac{1}{2}.$$

$$\mathbb{P}(Y_1 = 2) = \mathbb{P}(X = 2 | B_1) = \frac{1}{2}.$$

(ii) $Y_2 = X | B_2$

$$\mathbb{P}(Y_2 = 0) = \mathbb{P}(X = 0 | B_2) = \frac{1}{2}$$

$$\mathbb{P}(Y_2 = 1) = \frac{1}{2}, \quad \mathbb{P}(Y_2 = 2) = 0$$

In general, Y is a r. v. and hence

$$\sum_x \mathbb{P}(Y = x) = 1.$$

Def-n

$$\mathbb{E}(X | B) = \mathbb{E}(Y) = \sum_x x \, \mathbb{P}(Y = x) =$$

$$\sum_x x \frac{\mathbb{P}((X = x) \cap B)}{\mathbb{P}(B)} = \sum_x x \, \mathbb{P}(X = x | B_i)$$

In the above example

$$\mathbb{E}(Y_1) = 0 \times 0 + 1 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1.5$$

$$\mathbb{E}(Y_2) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} + 2 \times 0 = 0.5$$

The Theorem of Total Probability for Expectations

(TPTE) **Theorem** Let X be a discrete r.v., let $B_1, \ldots, B_n$ be a partition, and suppose that $\mathbb{P}(B_i) > 0$ for all $i = 1, 2, .., n$. Then

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X|B_i)\, P(B_i)$$

Proof. $\mathbb{E}(X) = \sum_x x\, \mathbb{P}(X = x)$.

Since B_i form a partition,

$$\mathbb{P}(X = x) = \sum_{i=1}^{n} P(X = x|B_i)\, \mathbb{P}(B_i)$$ (by the Total prob. Th-m). Hence

$$\mathbb{E}(X) = \sum_x x \sum_{i=1}^{n} \mathbb{P}(X = x|B_i) P(B_i)$$

$$= \sum_x \sum_{i=1}^{n} x\, \mathbb{P}(X = x|B_i)\, \mathbb{P}(B_i)$$

$$= \sum_{i=1}^{n} \sum_x x\, \mathbb{P}(X = x|B_i)\, P(B_i)$$

$$= \sum_{i=1}^{n} P(B_i) \underbrace{\sum_x x\, \mathbb{P}(X = x|B_i)}_{\mathbb{E}(X|B_i)}$$

$$= \sum_i \mathbb{E}(X|B_i)\, \mathbb{P}(B_i)$$

∎

Example. Alice and Bob play the following game. They roll a fair dice. If it comes up 1 or 2, Alice wins. If it comes up 3, Bob wins. If it comes up 4, 5, 6, they play again. What is the expected number of rolls? (They play until one of them wins).

Solution X = number of rolls.

X takes values $1, 2, 3, \dots, n, \dots$

and $\mathbb{E}(X) = \sum_{n=1}^{\infty} n\, \mathbb{P}(X=n)$ - this can be done. However, let us use the TPTE.

B_1 = the game ends after the 1^{st} roll

$B_2 = B_1^c$ = the game does not end after the 1^{st} roll.

$\mathbb{P}(B_1) = \frac{1}{2}$, $\mathbb{P}(B_2) = \frac{1}{2}$.

$$\mathbb{E}(X) = \mathbb{P}(B_1)\,\mathbb{E}(X|B_1) + \mathbb{P}(B_2)\,\mathbb{E}(X|B_2) -$$

by the TPTE. Next

$$\mathbb{E}(X|B_1) = 1 \times 1 = 1.$$

$$\mathbb{E}(X|B_2) = 1 + \mathbb{E}(X).$$ Hence

$$\mathbb{E}(X) = \frac{1}{2}\times 1 + \frac{1}{2}(1+\mathbb{E}(X)) = 1 + \frac{1}{2}\mathbb{E}(X)$$

$$\frac{1}{2}\mathbb{E}(X) = 1 \text{ and } \boxed{\mathbb{E}(X) = 2}.$$

Example

Geometric distr-n. Geom(p).

X = the # of trials until 1st success.

p = Prob of success in each trial.

$$\mathbb{P}(X=1) = p.$$

$$\mathbb{P}(X=2) = (1-p)p = qp, \quad q \stackrel{\text{def}}{=} 1-p$$

$$\mathbb{P}(X=3) = q^2 p$$

$$\mathbb{P}(X=n) = q^{n-1} p.$$

In the example, $p = \frac{1}{2}$, $q = \frac{1}{2}$. So, we proved that

$$\mathbb{E}(X) = 1\times\frac{1}{2} + 2\times\frac{1}{2^2} + 3\frac{1}{2^3} + \ldots + n\frac{1}{2^n} + \ldots = 2.$$

<u>Exercise</u>. Prove that if $X \sim \mathrm{Geom}(p)$, then $\mathbb{E}(X) = \frac{1}{p}$.

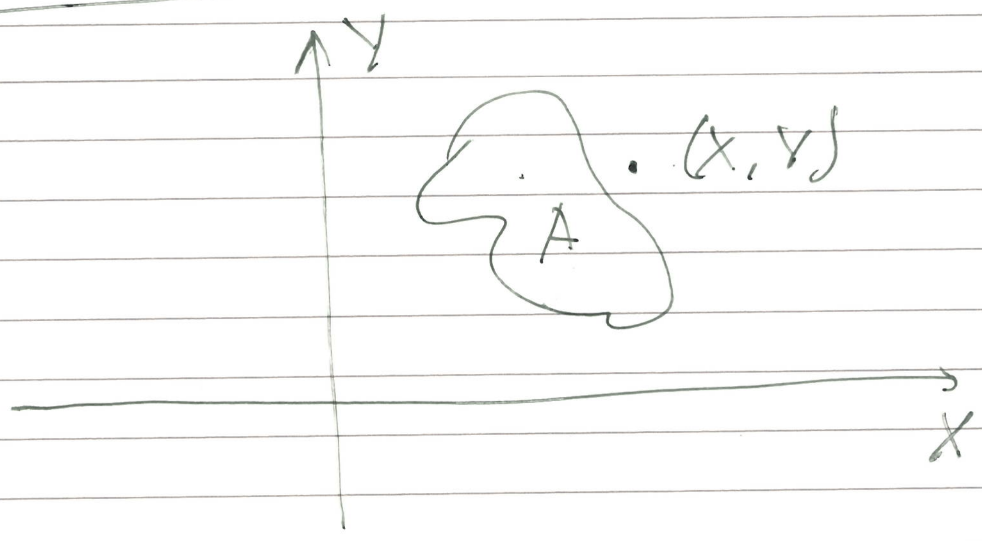
<u>Hint</u>. Use the TPTE.

<u>Jointly continuous r.v.'s</u>

<u>Def-n</u>. Let X, Y be r.v.'s. We say that X and Y are jointly continuous if there is a function $f_{X,Y}(x,y)$ such that

$$\mathbb{P}((X,Y) \in A) = \iint_A f_{X,Y}(s,t)\,ds\,dt,$$

where $A \subset \mathbb{R}^2$ is any subset of $\mathbb{R}^2$.



$f_{X,Y} = f_{X,Y}(\cdot,\cdot)$ is the joint probability density function (for X, Y), or joint pdf.

Properties of the joint pdf of X, Y.

(i) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(s,t)\,ds\,dt = 1$. $\quad \mathbb{P}(-\infty < X < \infty, -\infty < Y < \infty) = 1$

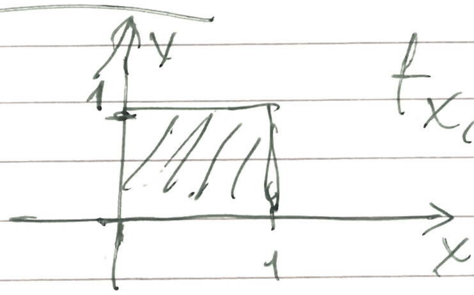
(ii) $f_{X,Y}(s,t) \geq 0$ for all s, t.

Example. Let X, Y have a joint pdf given by

$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

Find c.

Solution. We use (i)



$f_{X,Y}(s,t) = 0$ outside the square

Hence

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(s,t)\,ds\,dt = \int_0^1\int_0^1 f_{X,Y}(s,t)\,ds\,dt$$

$$= c\int_0^1\int_0^1 ds\,dt = c\int_0^1 \Big(\underbrace{\int_0^1 ds}_{s|_0^1 = 1}\Big) dt = c\int_0^1 dt = c \quad \boxed{\Rightarrow c = 1}$$

<u>Example</u>. Let X, Y be jointly cont. r.v.'s,

$$f_{X,Y}(s,t) = \begin{cases} e^{-s-t} & \text{if } s,t>0 \\ 0 & \text{otherwise.} \end{cases}$$

Find:

(i) $\mathbb{P}(3 < X < 4 \text{ and } 2 < Y < 5)$

(ii) $\mathbb{P}(-1 < X < 3 \text{ and } 2 < Y < 5)$

(iii) $\mathbb{P}(Y > X > 0)$.