'Chaos':

Dictionary definition:

'Complete disorder + confusion'

Maths / science definition:

The property of a complex system whose behaviour is so unpredictable as to appear random, owing to great sensitivity to small changes in conditions.

'Chaos Theory': The branch of maths that deal with complex systems whose behaviour is highly sensitive to slight changes in conditions, so that small alterations can give rise to strikingly great consequences.

Examples

Fluid flow: Drop 2 nearby bottles from an ocean liner at sea, they will end up far apart (possibly washing up onshore in opposite parts of the world)

- Drop 2 nearby sticks in "rapids" (fast flowing stream with rocks, etc), they'll move apart before long
- Release 2 nearby (helium) balloons on a windy day, they'll move far apart
- Weather prediction

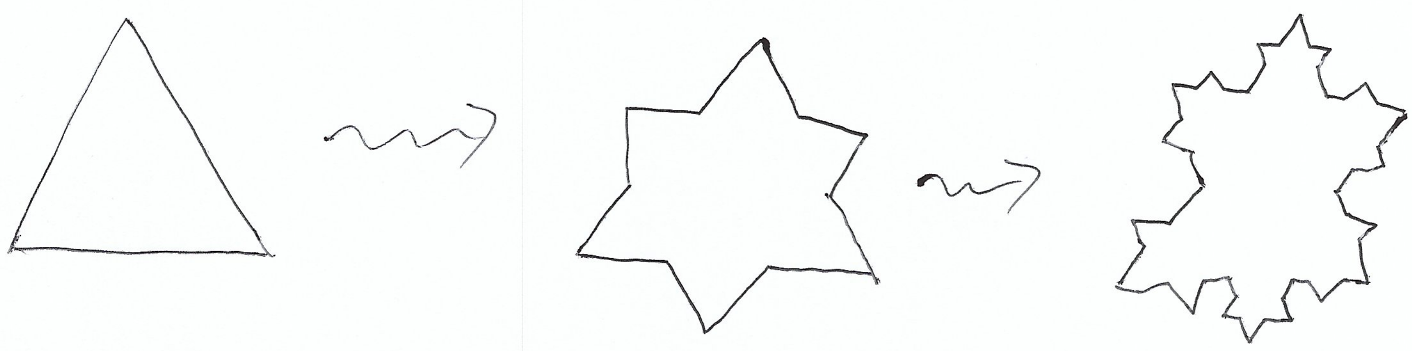
Chaos and Fractals

- A 'new' branch of maths, 40 to 50 years old
- Mathematicians working in Dynamical Systems started using computers in order to produce images of "chaotic" phenomena
- Fractal usually means a set of non-integer dimension (the word was first used by Mandelbrot in 1975)

from "fractional"

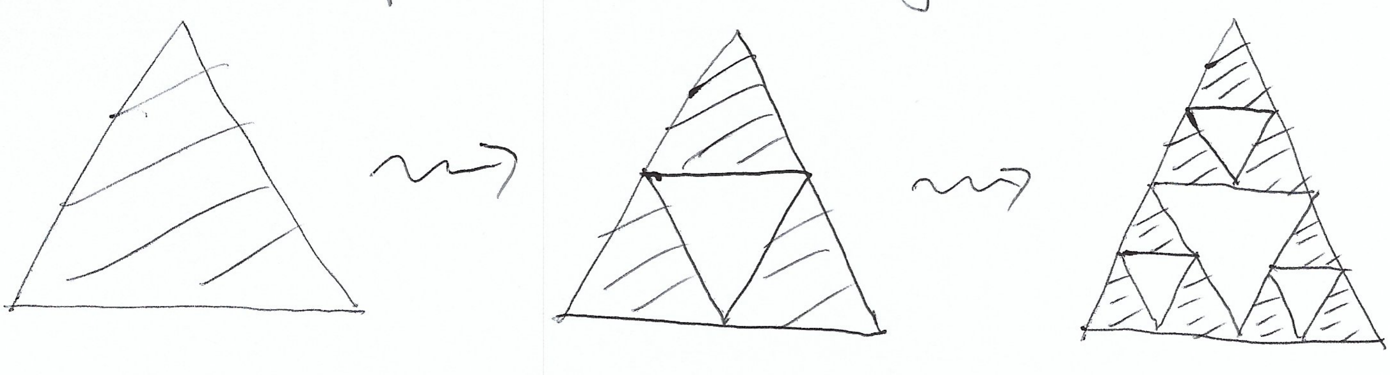
Example of Fractals

(i) von Koch snowflake



$\rightsquigarrow \dots \rightsquigarrow$ infinite process of refining the approximation to a geometric object which appears "in the limit" – this object has dimension $d \in (1, 2)$

(ii) Sierpinski Triangle



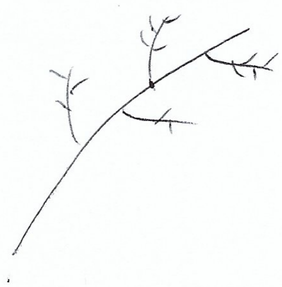
$\rightsquigarrow \dots \rightsquigarrow$

The "limit" is an object of dimension strictly between 1 and 2

(iii) Snowflakes

(iv) Other 'fractals' in nature

- wiggly rivers
- the coastline of Britain
- ferns



Chapter 1 Dynamical systems (on $\mathbb{R}$)

Let $f : \mathbb{R} \to \mathbb{R}$

This defines a recurrence relation

$$\boxed{x_{n+1} = f(x_n) \quad \text{for all } n \geqslant 0}$$

which is a dynamical rule for generating a sequence $(x_n)_{n=0}^{\infty}$ provided we have a starting point $x_0 \in \mathbb{R}$.

This is because

$$x_1 = f(x_0),$$

$$x_2 = f(x_1) = f(f(x_0)) = f^2(x_0)$$

$$x_3 = f(x_2) = f(f(x_1)) = f(f(f(x_0)))$$

$$= f^3(x_0)$$

You can think of x_n as representing "position at time n".

In particular, x_0 is our "initial position", i.e. "position at time zero".

<u>Note</u> The dynamical rule (namely $x_{n+1} = f(x_n) \quad \forall n \geq 0$) is <u>deterministic</u>.

<u>Defn</u> A point $x_0 \in \mathbb{R}$ is called a <u>fixed point</u> of f if

$$f(x_0) = x_0$$

For a given function f, define

$$\mathrm{Fix}(f) := \{ x_0 \in \mathbb{R} : f(x_0) = x_0 \}$$

to be the set of all fixed pts of f.

- A point $x_0 \in \mathbb{R}$ is said to be <u>periodic</u> (with period n) if

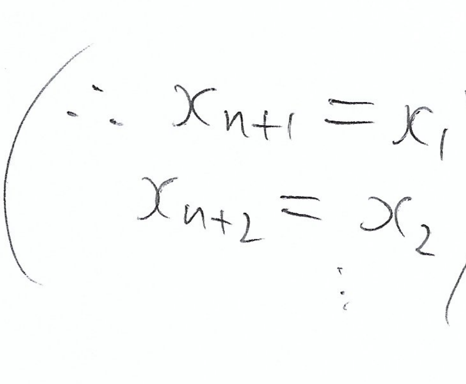
$$f^n(x_0) = x_0$$

i.e. $$\underbrace{(f \circ f \circ f \circ \dots \circ f)}_{n \text{ times}}(x_0) = x_0$$

i.e. $$\underbrace{f(f(f(\dots f}_{n \text{ times}}(x_0)))) = x_0$$

i.e. $x_n = x_0$ $\left(\begin{array}{l} \therefore x_{n+1} = x_1 \\ x_{n+2} = x_2 \\ \vdots \end{array} \right)$

$$\begin{tikzcd} x_0 \arrow[r] & x_1 \arrow[d] \\ x_3 \arrow[u] & x_2 \arrow[l] \end{tikzcd}$$



Warning: Here $f^n(x_0)$ denotes the n-fold composition $(\underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}})(x_0)$

In particular, $f^n(x_0)$ does NOT mean $f(x_0)^n$,

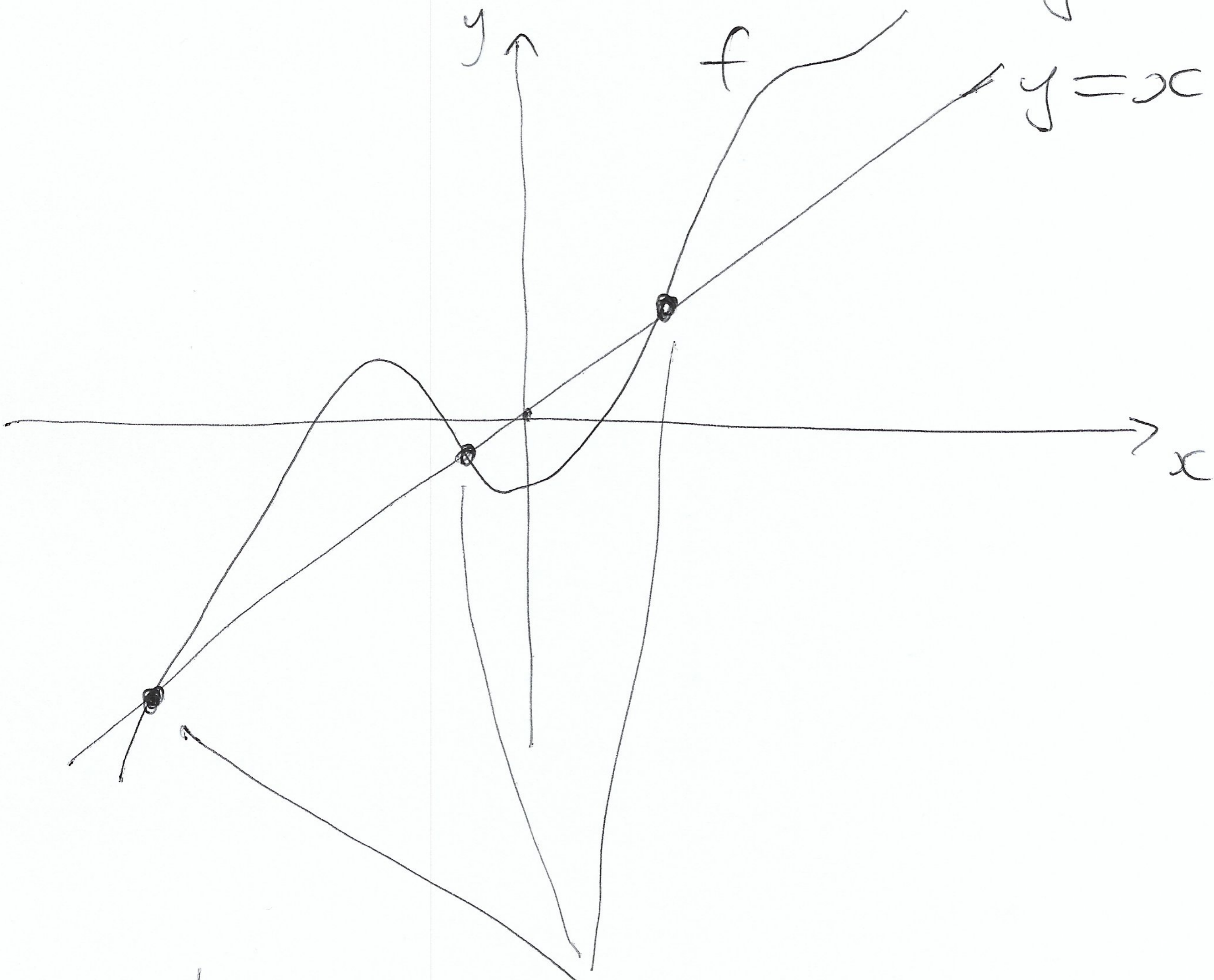
and $f^n(x_0)$ does NOT mean the n^{th} derivative $f^{(n)}(x_0)$

We denote the set of periodic pts of period n by

$$\mathrm{Per}_n(f) = \{x_0 \in \mathbb{R} : f^n(x_0) = x_0\}$$

Note:
- $\mathrm{Fix}(f) = \mathrm{Per}_1(f)$
- $\mathrm{Fix}(f^n) = \mathrm{Per}_n(f)$
- $\mathrm{Fix}(f) \subset \mathrm{Per}_n(f) \quad \forall n \geq 1$

Geometrically we can think of fixed points as intersections of the graph of f with the line $y = x$



These 3 points x are fixed points of the function f, i.e. they satisfy the equation $f(x) = x$

<u>Defn</u> The <u>orbit</u> of $x_0 \in \mathbb{R}$ under f is the set

$$\mathcal{O}(x_0) = \{x_0, f(x_0), f^2(x_0), \dots\}$$

$$= \{f^n(x_0) : n \geq 0\}$$

Equivalent definition

$$= \{\cancel{x} \; x_0, x_1, x_2, x_3, \dots\}$$

$$= \{x_n : n \geq 0\}$$

i.e. The orbit is a <u>set</u> (rather than a <u>sequence</u>)

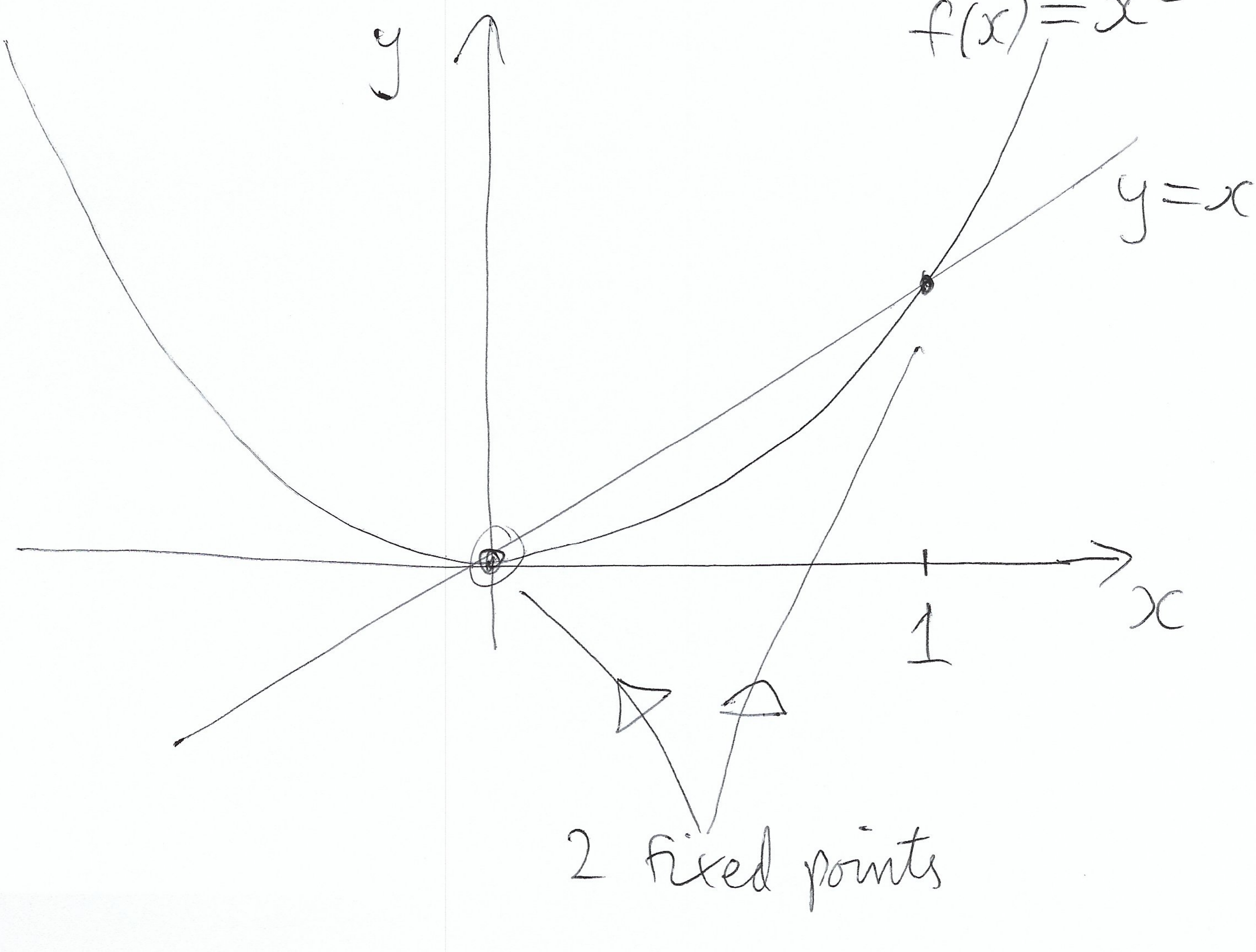
(though occasionally it is usefull to blur this distinction, and use "orbit" to describe the <u>sequence</u> $(x_n)_{n=0}^{\infty}$)

Example Let us define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$.

First, recall the recurrence relation:

$$x_{n+1} = f(x_n)$$

i.e. $$x_{n+1} = x_n^2.$$



The function f has 2 <u>fixed points</u>, namely 0 and 1

(i.e. the 2 solutions to the equation $x^2 = x$)

<u>Question</u>: If x_0 is <u>not</u> equal to 0 or 1, what is the "behavior" of the sequence $(x_n)_{n=0}^{\infty}$?

The orbit of any point is given by

$$x_0 \xrightarrow{f} x_1 \xrightarrow{f} x_2 \xrightarrow{f} x_3 \xrightarrow{f} x_4 \rightarrow \cdots$$

$$x_1 = x_0^2, \quad x_2 = x_1^2 = x_0^4, \quad x_3 = x_2^2 = x_1^4 = x_0^8, \quad x_4 = x_0^{16}$$

Observe that if $|x_0| > 1$ then the values of $x_n = f^n(x_0)$ $(= x_0^{2^n})$ will get large (in fact tend to ∞ as $n \to \infty$)

e.g. $x_0 = 3$

$$3 \xrightarrow{f} 9 \xrightarrow{f} 81 \xrightarrow{f} \cdots$$

On the other hand if $|x_0| < 1$ then the values of $x_n = f^n(x_0)$ will get smaller, in fact converge to 0 as $n \to \infty$:

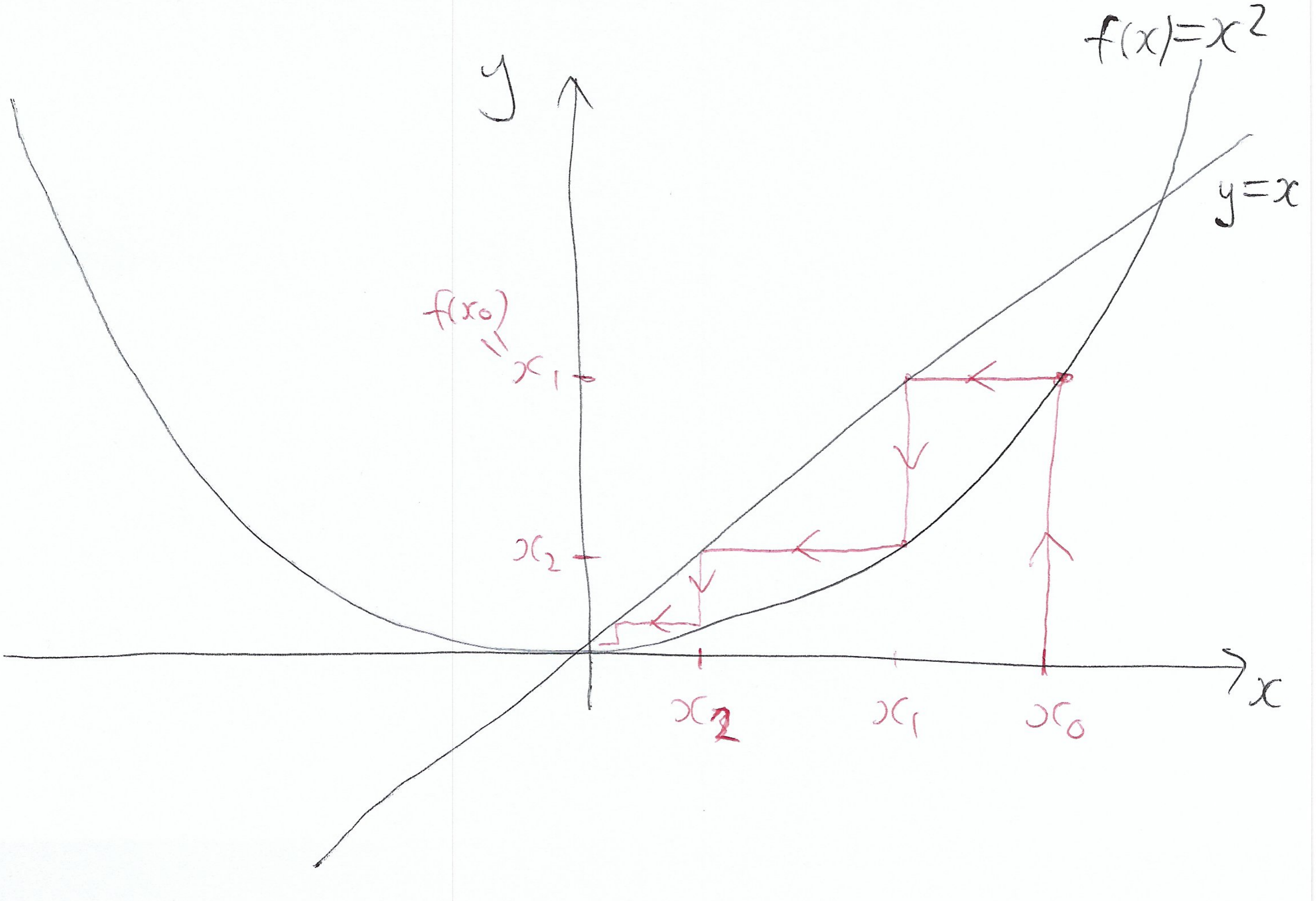
e.g. $x_0 = 0.1 = 1/10$

$$0.1 \xrightarrow{f} 0.01 \xrightarrow{f} 0.0001 \xrightarrow{f} \cdots$$

Summary

$$\lim_{n\to\infty} f^n(x_0) = \begin{cases} 0 & \text{if } |x_0| < 1 \\ \infty & \text{if } |x_0| > 1 \\ 1 & \text{if } |x_0| = 1 \end{cases}$$

Returning to the graph of $f(x)=x^2$, we can depict an orbit using a so-called <u>cobweb diagram</u>



Thus the fixed point 0 is "attracting" and if $|x_0| < 1$ then the sequence $(x_0, x_1, x_2, \ldots)$ tends to the fixed point.

By contrast, the fixed point 1 is "repelling". ~~and~~

Qu Does this function $f(x) = x^2$ have any periodic points (apart from the 2 fixed points at 0 and 1)?

Ans No!

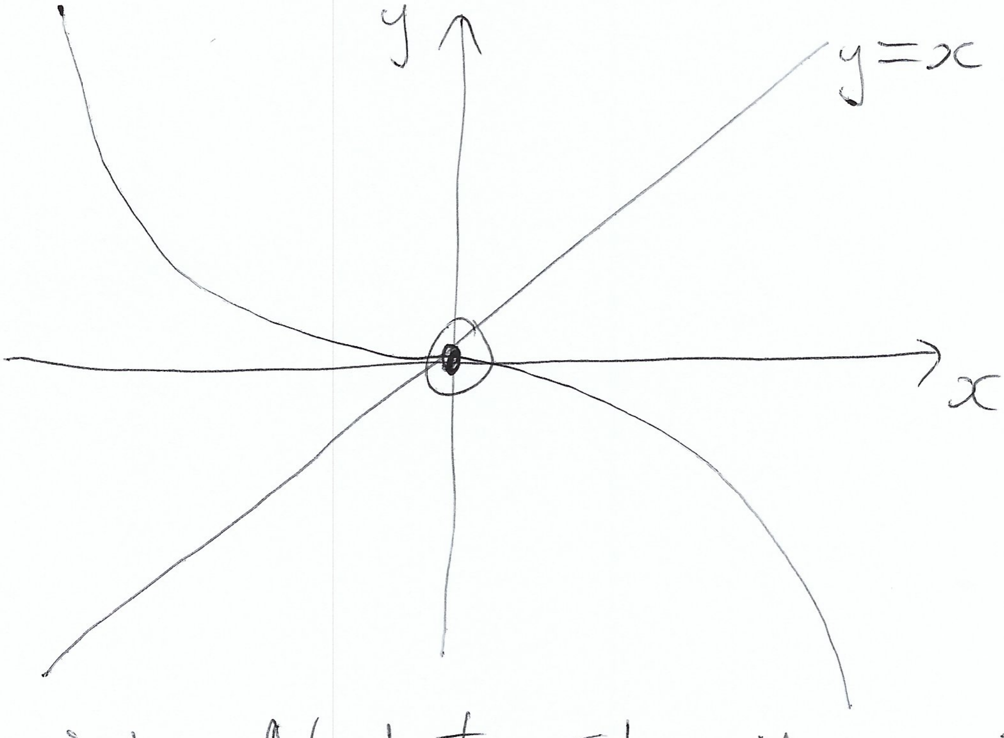
Example Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = -x^3$$

Examine fixed points and various orbits for this dynamical system f.

Recall the dynamical rule is given by

$$x_{n+1} = f(x_n) = -x_n^3$$



Fixed points: Need to solve the equation $f(x) = x$

i.e. $-x^3 = x$

i.e. $x^3 + x = 0$

i.e. $x(x^2+1) = 0$

The only (real) solution is $x = 0$

i.e. 0 is the only fixed point

What about other periodic points?

Let's examine the orbit of any point $x_0 \in \mathbb{R}$:

$$x_0 \xrightarrow{f} -x_0^3 \xrightarrow{f} x_0^9 \xrightarrow{f} -x_0^{27} \xrightarrow{f} \cdots$$

Notice that since the power goes up, the "size" of x_n is either growing or shrinking unless $x_0 = 1$ or $x_0 = -1$, so the only candidates for periodic points are $x_0 = 1$ and $x_0 = -1$

Let $x_0 = 1$. Then the orbit of x_0 under f

$$1 \xrightarrow{f} -1 \xrightarrow{f} 1 \xrightarrow{f} -1 \xrightarrow{f}$$

i.e.

$$\begin{tikzcd} 1 \arrow[r, bend left, "f"] & -1 \arrow[l, bend left, "f"] \end{tikzcd}$$

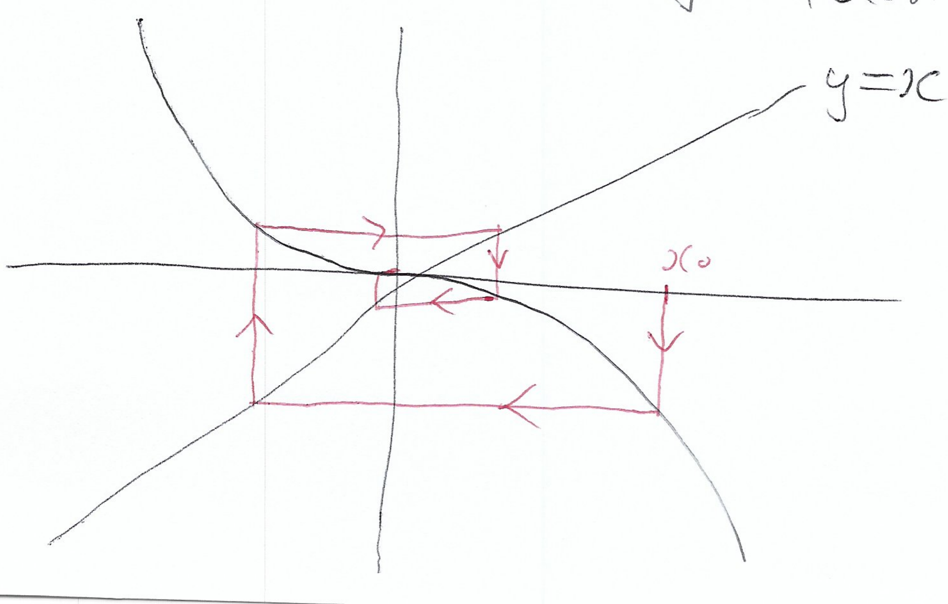
Thus 1 is a point of period 2,
and -1 is a point of period 2.

We can say that $\{1, -1\}$ is an orbit of period 2.

What about other $x_0 \in \mathbb{R}$?

We notice that $|f^n(x_0)| \to 0$ as $n \to \infty$ if $|x_0| < 1$,
and $|f^n(x_0)| \to \infty$ as $n \to \infty$ if $|x_0| > 1$.

Thus 0 is an "attracting" fixed point



Definition Given a periodic point x_0, let m be the smallest natural number (i.e. smallest strictly positive integer) such that $f^m(x_0) = x_0$. Then m is called the least period (or prime period) of x_0.

Example If $f(x) = -x^3$, then $x_0 = 1$ has period 2 (but also period 4, 6, 8, ...) but its least period is 2.

Definition If x_0 has least period m, then the orbit $\{x_0, f(x_0), \dots, f^{m-1}(x_0)\}$ is often called a m-cycle (or simply a cycle).

Question If x_0 has least period 2, can it also have period 7?

Answer No: If $x_0 = f^7(x_0)$

$$= f(f^6(x_0))$$

$$= f(x_0),$$

which says that x_0 is a fixed point, which contradicts the fact that its least period is equal to 2.

Lemma Let $x \in \mathrm{Per}_n(f) = \{x_0 : f^n(x_0) = x_0\}$, and suppose x has least period k.

Then k must be a divisor of n.

(i.e. k must be a factor of n)

Proof Suppose k is not a divisor of n.

So $n > k$.

So let us write $n = qk + r$

(where $q \in \mathbb{N}$, and $r \in \mathbb{Z}$ with $1 \leq r \leq k-1$)

Now
$$
\begin{aligned}
f^{qk}(x) &= f^{(q-1)k}(f^k(x)) \\
&= f^{(q-1)k}(x) \\
&= f^{(q-2)k}(f^k(x)) \\
&= f^{(q-2)k}(x) \\
&\;\;\vdots \\
&= f^k(x) \\
&= x \qquad (*)
\end{aligned}
$$

So
$$
\begin{aligned}
f^n(x) &= f^{qk+r}(x) \\
&= f^r(f^{qk}(x)) \\
&= f^r(x)
\end{aligned}
$$

by $(*)$

$\neq x$ since $1 \leq r \leq k-1 < k$
and k is the <u>least</u> period

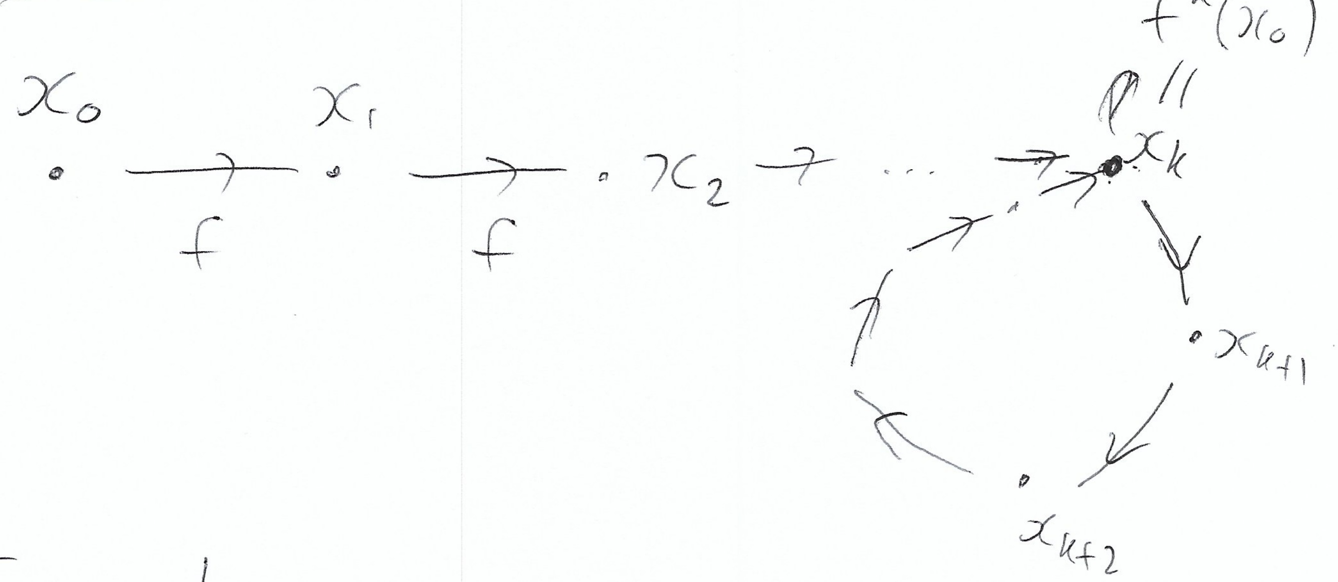
But $f^n(x) \neq x$ means that
$x \notin Per_n(f)$, a <u>contradiction</u>, as required.

$\square$

<u>Definition</u> We say that a point x_0 is <u>eventually periodic</u> (or <u>pre-periodic</u>) <u>of period n</u> if there exists some integer $k \geq 0$ such that $\underbrace{f^k(x_0)}_{= x_k}$ is periodic of period n.

$$\left(\text{i.e. } x_{k+n} = x_k \right.$$
$$\left. \text{i.e. } f^n(f^k(x_0)) = f^k(x_0) \right)$$

Schematic diagram :



Example For $f(x) = x^2$, the point $x_0 = -1$ is eventually periodic of period 1 (we could also say it is an eventually fixed point)

since $f(-1) = 1$, and 1 is a fixed point.

Example Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 2$.

Qu Does f have any fixed points? If so, what are they?

Fixed points are solutions of the equation $f(x) = x$

i.e. $x^2 - 2 = x$

i.e. $x^2 - x - 2 = 0$

i.e. $(x-2)(x+1) = 0$

i.e. $x = 2$ and $x = -1$ are the only fixed points of f