Queen Mary
University of London

MTH5104: Convergence and Continuity 2023–2024
Problem Sheet 1 (Demon Games)

Consider the following five statements:

(a) $\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}\ :\ n = m^2$.

(b) $\forall a \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}, |x - a| < \delta\ :\ |3x - 3a| < \varepsilon$.

(c) $\forall x \in \mathbb{Q}\ \forall y \in \mathbb{Q}, y \neq x\ \exists z \in \mathbb{Q}\ :\ (x < z < y)$ or $(y < z < x)$.

(d) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R}, |x| < \delta\ :\ x^2 < \varepsilon$.

(e) $\forall N \in \mathbb{N}\ \exists n \in \mathbb{N}, n \geq N\ \exists m \in \mathbb{N} : n = m!$.

Questions.

1. Write down the Demon games corresponding to (a)-(e)

2. Write down the *negation* of each of the statements (a)-(e).

3. For each of the statements (a)-(e), write down two *trial games* for the corresponding Demon game: one in which we win, one in which the Demon wins.

   (That is, write down two sequences of legal moves and state who wins if those are the moves played. You are *not* asked to find a winning strategy.)

4. For each of (b)–(e), give a *winning strategy* for the corresponding Demon game. Briefly explain why the strategy works. Then write down this winning strategy as a mathematical proof.

5. **Challenge.** Find a *winning strategy* for the Demon Game corresponding to the expression:

$$\forall x \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in \mathbb{R}, |x - y| < \delta\ :\ |x^2 - y^2| < \varepsilon.$$