$f_x(t)$ and its relation to force of mortality

$$f_x(t) = \frac{d}{dt} F_x(t) = \frac{d}{dt} P[T_x \leq t]$$

$$= \lim_{h \to 0} \frac{P[T_x \leq t+h] - P[T_x \leq t]}{h}$$

if we use T instead of T_x

$$f_x(t) = \lim_{h \to 0} \frac{P[T \leq x+t+h \mid T > x] - P[T \leq x+t \mid T > x]}{h}$$

now expand the two conditional probabilities

$$f_x(t) = \lim_{h \to 0} \frac{1}{h} \left[\frac{P[T \leq x+t+h]}{P[T > x]} - \frac{P[T \leq x+t]}{P[T > x]} \right]$$

now $P[T > x]$ is $S(x)$ the survival function

$$\therefore f_x(t) = \lim_{h \to 0} \frac{P[T \leq x+t+h] - P[T \leq x+t]}{h.\, S(x)}$$

now multiply top and bottom by $S(x+t)$

$$f_x(t) = \lim_{h \to 0} \frac{S(x+t)}{S(x)} \left(\frac{P[T \leq x+t+h] - P[T \leq x+t]}{h.\, S(x+t)} \right)$$

This does not vary with h so can come outside the limit

$S(x+t) = P[T > x+t]$

$$\therefore\ f_x(t) = \frac{S(x+t)}{S(x)} \lim_{h \to 0} \frac{1}{h} \cdot \frac{P[T \le x+t+h] - P[T \le x+t]}{P[T > x+t]}$$

$$f_x(t) = \frac{S(x+t)}{S(x)} \lim_{h \to 0} \frac{1}{h} \left(\frac{P[T \le x+t+h]}{P[T > x+t]} - \frac{P[T \le x+t]}{P[T > x+t]} \right)$$

These two can be expressed as conditional probabilities

$$f_x(t) = \frac{S(x+t)}{S(x)} \lim_{h \to 0} \frac{1}{h} \Big\{ P[T \le x+t+h \mid T > x+t] - P[T \le x+t \mid T > x+t] \Big\}$$

this second conditional probability is zero

leaving

$$f_x(t) = \underbrace{\frac{S(x+t)}{S(x)}} \ \underbrace{\lim_{h \to 0} \frac{1}{h} P[T \le x+t+h \mid T > x+t]}$$

which is $S_x(t)$ or ${}_tp_x$

this is the force of mortality μ_{x+t}

so

$$f_x(t) = {}_tp_x \, \mu_{x+t}$$