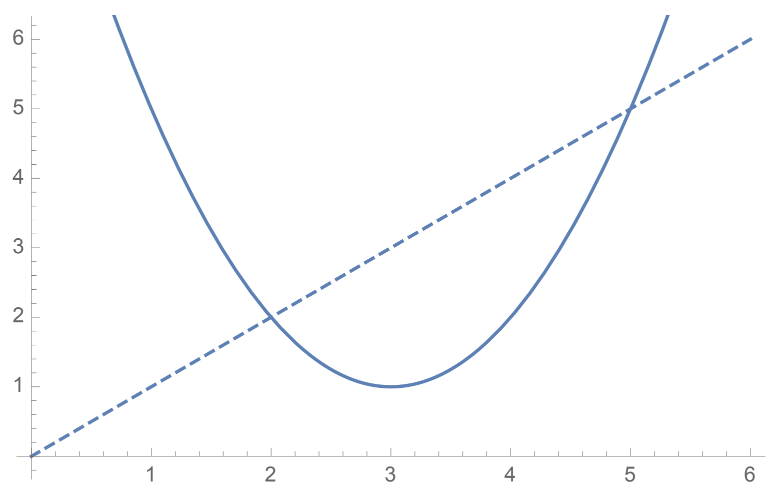
# MTH6107 Chaos & Fractals

Solutions 1

(A) Suppose the map $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - 6x + 10$.

**Exercise 1.** Draw the graph of the map $f$, and determine all its fixed points. Determine which of these points are attracting and which of these points are repelling.

Fixed points satisfy $f(x) = x$, so $x^2 - 6x + 10 = x$, so $x^2 - 7x + 10 = 0$, so $x = 2$ and $x = 5$ are the fixed points.




Note that $f'(x) = 2x - 6$, so the multipliers at the fixed points are $f'(2) = -2$, and $f'(5) = 4$; both these multipliers have modulus strictly larger than 1, so both fixed points are *repelling*.

**Exercise 2.** For the map $f$, determine an eventually fixed point which is not a fixed point.

The point 1 is an example of an eventually fixed point (there are infinitely many other examples), since $f(1) = 5$.

**Exercise 3.** Draw a graph of the map $f^2$. Determine all the points of prime period 2 of $f$. Determine which of these points are attracting and which of these points are repelling.

Points of period 2 satisfy $f^2(x) = x$, so

$$(x^2 - 6x + 10)^2 - 6(x^2 - 6x + 10) + 10 = x\,,$$

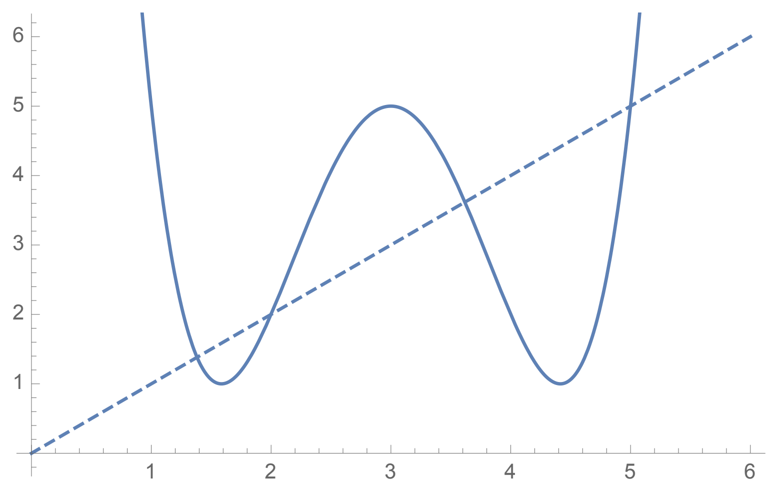
in other words

$$x^4 - 12x^3 + 50x^2 - 85x + 50 = 0\,.$$

We know that the two fixed points also have period 2, so we know that $x^2 - 7x + 10 = (x-2)(x-5)$ is a factor of the above righthand side, so we can factorise it to give the equation

$$(x-2)(x-5)(x^2 - 5x + 5) = 0\,,$$

so the two roots of $x^2 - 5x + 5$, namely $(5 \pm \sqrt{5})/2$, are points of prime period 2.




The multiplier of the 2-cycle $\{(5+\sqrt{5})/2, (5-\sqrt{5})/2\}$ is equal to

$$f'((5+\sqrt{5})/2)f'((5-\sqrt{5})/2)\,,$$

and since $f'(x) = 2x - 6$ then $f'((5+\sqrt{5})/2) = -1+\sqrt{5}$ and $f'((5-\sqrt{5})/2) = -1-\sqrt{5}$ so

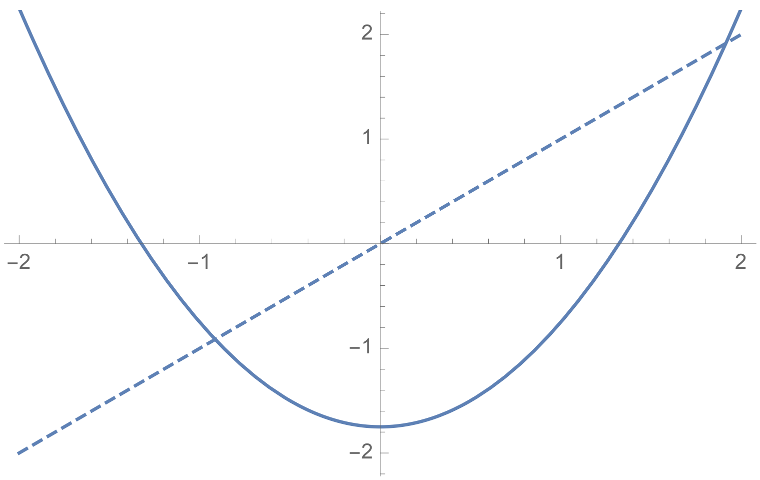
$$|f'((5+\sqrt{5})/2)f'((5-\sqrt{5})/2)| = |1-5| = |-4| > 1\,,$$

therefore the 2-cycle is *repelling*.

(B) Now suppose the map $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 - 7/4$.

**Exercise 4.** Draw the graph of the map $f$, and determine all its fixed points. Determine which of these points are attracting and which of these points are repelling.

Fixed points satisfy $f(x) = x$, so $x^2 - x - 7/4 = 0$, so $x = (1+\sqrt{8})/2$ and $x = (1-\sqrt{8})/2$ are the two fixed points.




The multipliers at these fixed points are $f'((1+\sqrt{8})/2) = 1+\sqrt{8}$ and $f'((1-\sqrt{8})/2) = 1-\sqrt{8}$, both of which have modulus strictly larger than 1, so both fixed points are *repelling*.

**Exercise 5.** For the map $f$, determine an eventually fixed point which is not a fixed point.

One eventually fixed point is $-(1+\sqrt{8})/2$, since $f(-(1+\sqrt{8})/2) = (1+\sqrt{8})/2$.

**Exercise 6.** Draw a graph of the map $f^2$. Determine all the points of prime period 2 of $f$. Determine which of these points are attracting and which of these points are repelling.

Points of period 2 satisfy $f^2(x) = x$, so

$$(x^2 - 7/4)^2 - 7/4 = x\,,$$

which can be written as

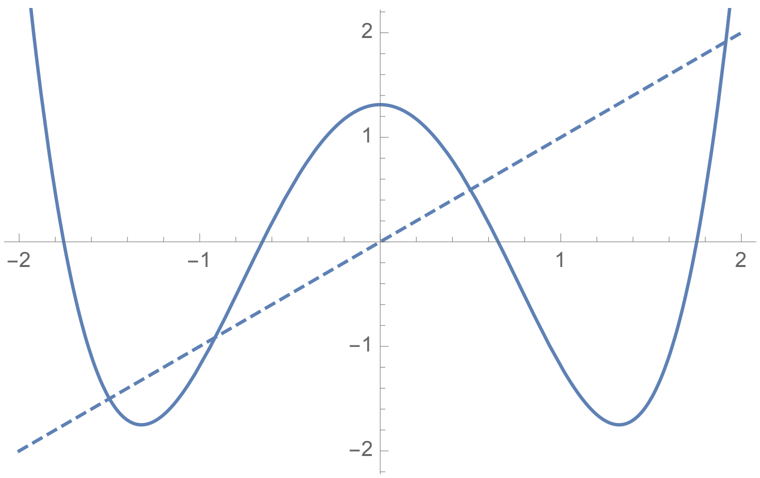
$$(x^2 - x - 7/4)(x - 1/2)(x + 3/2) = 0\,,$$

so the points of prime period 2 are $1/2$ and $-3/2$.

Now $f'(1/2) = 1$ and $f'(-3/2) = -3$, so the multiplier for the 2-cycle $\{1/2, -3/2\}$ is

$$f'(1/2)f'(-3/2) = -3\,,$$

which is greater than 1 in modulus, so this 2-cycle is *repelling*.

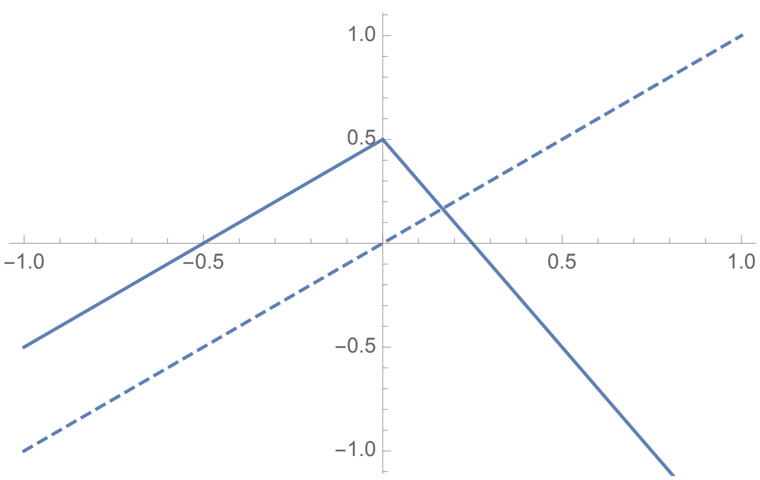



(C) Now suppose the map $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x + 1/2 & \text{for } x < 0 \\ -2x + 1/2 & \text{for } x \geq 0\,. \end{cases}$$

**Exercise 7.** Draw the graph of the map $f$, and determine all its fixed points. Determine which of these points are attracting and which of these points are repelling.

Fixed points satisfy $f(x) = x$. There are no such points $x < 0$, since $x + 1/2 = x$ has no solutions. There is one solution $x > 0$, namely the solution to $-2x + 1/2 = x$, namely $x = 1/6$ is the unique fixed point.




Since $|f'(1/6)| = |-2| = 2 > 1$ then this fixed point is *repelling*.

**Exercise 8.** For the map $f$, determine an eventually fixed point which is not a fixed point.

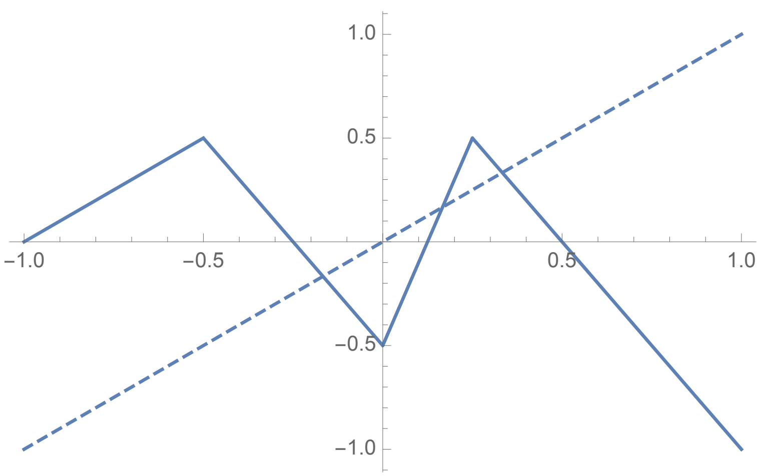
The point $-1/3 = 1/6 - 1/2$ satisfies $f(-1/3) = 1/6$, so $-1/3$ is an eventually fixed point. Note, for example, that more generally $1/6 - n/2$ is an eventually fixed point for all natural numbers $n$.

**Exercise 9.** Draw a graph of the map $f^2$. Determine all the points of prime period 2 of $f$. Determine which of these points are attracting and which of these points are repelling.

Period-2 points $x$ satisfy $f^2(x) = x$. If $x \leq -1/2$ then $f^2(x) = x + 1 \neq x$, so such an $x$ cannot have period 2. If $x \in (-1/2, 0)$ then

$$f^2(x) = f(x + 1/2) = -2(x + 1/2) + 1/2 = -2x - 1/2\,,$$

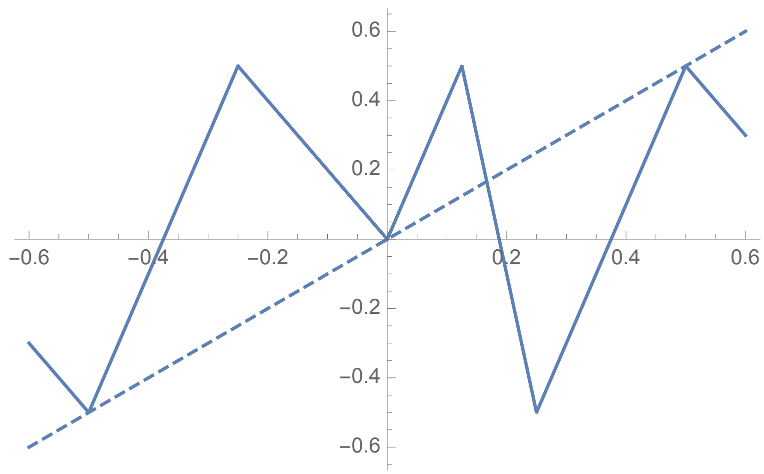
so $f^2(x) = x$ means $-1/2 = 3x$, so $x = -1/6$. Note that $f(-1/6) = 1/3$, so $\{-1/6, 1/3\}$ is a 2-cycle. In fact it is easily seen that it is the only 2-cycle (see e.g. that by inspection of the graph of $f^2$ there are 3 solutions to $f^2(x) = x$, namely the fixed point $1/6$, and the two period-2 points $-1/6$ and $1/3$).




The 2-cycle $\{-1/6, 1/3\}$ is *repelling*, since $|(f^2)'(-1/6)| = |-2| > 1$.

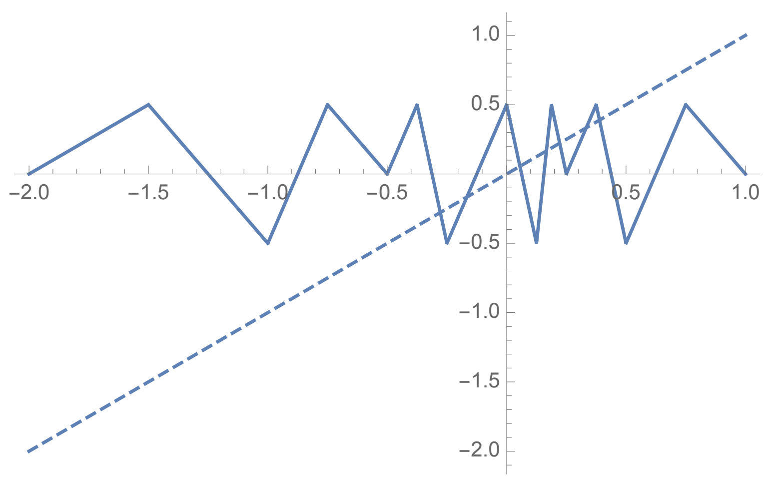
**Exercise 10.** For the map $f$, determine all its points of prime period 3.

The points of period 3 are solutions to $f^3(x) = x$ (see below for a graph of $f^3$), and we can calculate that, in addition to the fixed point at $1/6$, the only such points are at $-1/2$, $0$, and $1/2$, so these 3 points are of prime period 3, and constitute a 3-cycle.




**Exercise 11.** For the map $f$, determine all its points of prime period 4.

The points of period 4 are solutions to $f^4(x) = x$ (see below for a graph of $f^4$), and we can calculate that, in addition to the fixed point at $1/6$, and the points $-1/6, 1/3$ of prime period 2, the only such points are at $-5/18$, $1/18$, $2/9$, and $7/18$, so these 4 points are of prime period 4, and constitute a 4-cycle.



**Exercise 12.** Is it the case that $f$ has a point of prime period $n$ for every $n \in \mathbb{N}$?

Indeed $f$ has a point of prime period $n$ for every $n \in \mathbb{N}$, by Sharkovskii's Theorem, since $f$ is continuous and has a point of prime period 3.

**Exercise 13.** Can you guess (or even prove) a formula for the number of points of period $n$ for the map $f$?

Let $P_n$ denote the number of points of period $n$. Exercise 7 gives $P_1 = 1$, exercise 9 gives $P_2 = 1 + 2 = 3$, exercise 10 gives $P_3 = 1 + 3 = 4$, and exercise 11 gives $P_4 = 1 + 2 + 4 = 7$.

These exercises may *suggest* that there are $n$ points of prime period $n$ for each $n \in \mathbb{N}$ (and hence that $P_n$ equals the sum of the divisors of $n$). However this is *false*, and it fails at $n = 5$: you can check that there are *two* orbits of prime period 5, so 10 points of prime period 5, therefore $P_5 = 1 + 10 = 11$ (the 11 solutions to the equation $f^5(x) = x$ are visible in the graph of $f^5$ below).

In fact the sequence $(P_n)_{n\geq 1}$ begins as $1, 3, 4, 7, 11, 18, 29, 47, 76, 123, \ldots$, and it can be shown that $P_n$ is the solution of the difference equation $P_{n+2} = P_n + P_{n+1}$ with initial values $P_1 = 1$, $P_2 = 3$ (this is reminiscent of the construction of the Fibonacci sequence). An alternative formula for $P_n$ is as the trace (the sum of the diagonal entries) of the $n^{th}$ power of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.