1.

(a) The chain is irreducible so (by Theorem 4.8) it has a unique equilibrium distribution.

However it does not have a limiting distribution essentially because there is some periodic behaviour; we alternate between state 1 and the set of states $\{2,3\}$. So

$$\begin{aligned}\mathbb{P}(X_t = 1 \mid X_0 = 1) &= 0 \text{ whenever } r \text{ is odd}\\ \mathbb{P}(X_t = 2 \mid X_0 = 1) &= 0 \text{ whenever } r \text{ is even}\\ \mathbb{P}(X_t = 3 \mid X_0 = 1) &= 0 \text{ whenever } r \text{ is even}\end{aligned}$$

So there is no way P^t can tend to a limit of the correct form as $t \to \infty$.

(b)
- If we start in state 1 we are certain to return in exactly 2 steps. So $\mathbb{P}(R_1 = 2) = 1$ and $\mathbb{E}(R_1) = 2$.
- If we start in state 2 then we may return at any even number of steps and

$$\mathbb{P}(R_2 = 2k) = p_{2,1}\left(p_{1,3}p_{3,1}\right)^{k-1} p_{1,2} = \left(\frac{4}{5}\right)^{k-1}\frac{1}{5}.$$

  So $\frac{1}{2}R_2$ has a geometric distribution with parameter $1/5$ and $\mathbb{E}(R_2) = 10$.
- Similarly, if we start in state 3 then we may return at any even number of steps and

$$\mathbb{P}(R_3 = 2k) = p_{3,1}\left(p_{1,2}p_{2,1}\right)^{k-1} p_{1,3} = \left(\frac{1}{5}\right)^{k-1}\frac{4}{5}.$$

  So $\frac{1}{2}R_3$ has a geometric distribution with parameter $4/5$ and $\mathbb{E}(R_2) = 5/2$.

(c) We saw in lectures (Theorem 5.3) that the equilibrium distribution is given by $w_k = \frac{1}{\mathbb{E}(R_k)}$ so it is $\begin{pmatrix} 1/2 & 1/10 & 2/5 \end{pmatrix}$.

(d) Finding the equilibrium distribution by solving $\mathbf{w}P = \mathbf{w}$ gives equations:

$$\begin{aligned} w_1 &= w_2 + w_3\\ w_2 &= \tfrac{1}{5}w_1\\ w_3 &= \tfrac{4}{5}w_1\end{aligned}$$

So $\mathbf{w} = \begin{pmatrix} w_1 & \frac{1}{5}w_1 & \frac{4}{5}w_1 \end{pmatrix}$ and because $\mathbf{w}$ is a probability vector we must have $2w_1 = 1$. This give the same answer.

I hope this question helped see the connection between the equilibrium distribution and the expected first return times. However, finding the equilibrium distribution is typically easier to do directly (as in part (d)) rather than by the method in parts (b-c). So we usually use Theorem 5.3 in the opposite direction (that is finding $\mathbb{E}(R_s)$ from the equilibrium distribution rather than vice versa).

2. Even though the question didn't ask you to draw the transition graph it's a good idea to do this to help visualise the chain. Do this now if you haven't already done so.

(a) By considering the transition graph we see that

$$f_0^{(t)} = p_{0,1}p_{1,2}p_{2,3}\dots p_{t-2,t-1}p_{t-1,0} = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\cdots\left(\frac{t-1}{t}\right)\left(\frac{1}{t+1}\right) = \frac{1}{t(t+1)}.$$

(Strictly speaking we should check the $t = 1$ case since we implicitly assumed that $t \geqslant 2$ in the above calculation. Happily $f_0^{(1)} = p_{0,0} = 1/2$ so the same formula works. Well done if you spotted this subtlety.)

Now

$$f_0 = \sum_{t=1}^{\infty} f_0^{(t)} = \sum_{t=1}^{\infty} \frac{1}{t(t+1)}$$

We need to decide whether this infinite sum is equal to 1. Now

$$\sum_{t=1}^{n} \frac{1}{t(t+1)} = \sum_{t=1}^{n}\left(\frac{1}{t} - \frac{1}{t+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

which tends to 1 as $n \to \infty$. So $f_0 = \sum_{t=1}^{\infty} \frac{1}{t(t+1)} = 1$. So state 0 is recurrent by definition.

(b) Using the expression for $f_0^{(t)}$ above

$$\mathbb{E}(R_0) = \sum_{t=1}^{\infty} t f_0^{(t)} = \sum_{t=1}^{\infty} \frac{1}{t+1} = \infty$$

and so the chain is null recurrent.

(c) The chain is irreducible (that is every state is in the same communicating class) and null recurrence is a class property. This means that every state shows the same behaviour. Since state 0 is null recurrent, state 1000 is also null recurrent.

(d) The transition graph for this process has the same shape as that for the success runs chain. Only the numerical values of the positive probabililties have changed, You could think of this process as being like the success runs chain except that the longer the current run of successes is the more likely it is to be extended. For instance a football team may be more confident during a long run of wins and hence more likely to win the next game.

3.

(a) If $\mathbf{w} = \begin{pmatrix} w_0 & w_1 & w_2 & \dots \end{pmatrix}$ is an equilibrium distribution then it satisfies $w_k = \sum_{i=0}^{\infty} w_i p_{i,k}$ (this is exactly the infinite analogue of the equation we get from $\mathbf{w}P = \mathbf{w}$). So in this case:

$$\begin{aligned} w_0 &= \tfrac{2}{3} w_1 \\ w_1 &= w_0 + \tfrac{2}{3} w_2 \\ w_2 &= \tfrac{1}{3} w_1 + \tfrac{2}{3} w_3 \\ &\vdots \end{aligned}$$

and in general $w_k = \frac{1}{3} w_{k-1} + \frac{2}{3} w_{k+1}$ for $k \geqslant 2$.

Solving the first few of these we get:

$$\begin{aligned} w_1 &= \tfrac{3}{2} w_0 \\ w_2 &= \tfrac{3}{2} w_1 - \tfrac{3}{2} w_0 = \tfrac{3}{4} w_0 \\ w_3 &= \tfrac{3}{2} w_2 - \tfrac{1}{2} w_1 = \tfrac{3}{8} w_0 \end{aligned}$$

This suggests a guess that the solution is $w_k = \frac{3}{2^k} w_0$ for all $k \geqslant 1$. Let's prove this by (strong) induction. We already checked it for $k = 1, 2$ (the base cases). Suppose that $k \geqslant 2$ and that $w_k = \frac{3}{2^k} w_0$ and $w_{k-1} = \frac{3}{2^{k-1}} w_0$. We have

$$\begin{aligned} w_{k+1} &= \tfrac{3}{2} w_k - \tfrac{1}{2} w_{k-1} \\ &= \tfrac{3}{2} \tfrac{3}{2^k} w_0 - \tfrac{1}{2} \tfrac{3}{2^{k-1}} w_0 && \text{(by the induction hypothesis)} \\ &= \left(\tfrac{9}{2^{k+1}} - \tfrac{3}{2^k} \right) w_0 \\ &= \tfrac{3}{2^{k+1}} w_0 \end{aligned}$$

as required.

Finally, for an equilibrium distribution we need $\sum_{i=0}^{\infty} w_i = 1$ so

$$1 = w_0 + w_1 + w_2 + \ldots = w_0 + w_0 \sum_{i=1}^{\infty} \frac{3}{2^k} = 4w_0$$

so the equilibrium distribution is

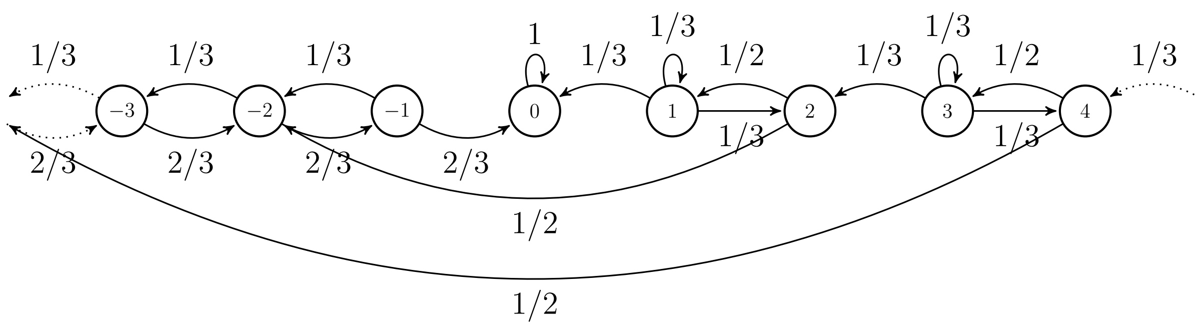
$$w_0 = \frac{1}{4}$$
$$w_k = \frac{3}{2^{k+2}} \quad \text{for } k \geqslant 1$$

(b) The chain has a unique equilibrium distribution so by Theorem 5.3 it is positive recurrent.

(c) This is asking for $\mathbb{E}(R_0)$. By Theorem 5.3 this is $\frac{1}{w_0}$ so $\mathbb{E}(R_0) = 4$.

4.

(a) The transition graph can look a bit of a mess. Drawing it systematically and thinking about how you arrange the states and arrows on the page helps a bit.



(b) From the graph the communicating classes are:

$$\{0\}, \quad \{-1, -2, -3, \ldots\}, \quad \{1, 2\}, \quad \{3, 4\}, \quad \ldots, \quad \{2k-1, 2k\}, \ldots$$

(c)

- State 0 is absorbing so $\{0\}$ is positive recurrent.
- State 1 is transient because if we go from 1 to 0 (which happens with probability 1/3 then we never return). So the states in $\{1, 2\}$ are transient.

- Similarly state $2k-1$ is transient because if we go from $2k-1$ to $2k-2$ (which happens with probability $1/3$) then we never return. So states in $\{2k-1, 2k\}$ are transient.
- All that's left are the negative integer states. These are also transient. From state -1 we could go to state 0 with probability $2/3$ and from there we can never return. So state -1 is transient. So every state in this communicating class (that is all the negative integer states) is transient.

5.

(a) If i and j are in the same communicating class then there exist $r, s > 0$ with $p_{ij}^{(r)} > 0$ and $p_{ji}^{(s)} > 0$. Suppose that i is 2-periodic, then since

$$p_{ii}^{(r+s)} \geqslant p_{ij}^{(r)} p_{ji}^{(s)} > 0$$

we have that $r+s$ must be even.

We want to prove that j is 2-periodic.

Suppose, for a contradition, that it is not. So there exists an odd number t such that $p_{jj}^{(t)} > 0$. Hence,

$$p_{ii}^{(r+t+s)} \geqslant p_{ij}^{(r)} p_{jj}^{(t)} p_{ji}^{(s)} > 0.$$

Since t is odd and $r+s$ is even $r+t+s$ is odd. This contradicts the 2-periodicity of i. We conclude that j is 2-periodic.

(b) Yes. A simple (and completely deterministic) example would be the chain with $S = \{1, 2, 3\}$ and transition matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(c) This is not possible. If a state has a loop then it is not 2-periodic (because $p_{ss}^{(1)} > 0$). We showed in part (a) that 2-periodicity is a class property so an irreducible chain cannot contain both a 2-periodic state and a state which is not 2-periodic.

Please let me know if you have any comments or corrections

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