1.

(a) There is no way to get from state 2 to state 1 so $p_{2,1}^{(t)} = 0$ for all $t \geqslant 0$. Hence the chain is not irreducible so it is certainly not regular.

(b) Suppose we start in state 1. We will eventually leave state 1 and then we will never return. So eventually, the chain will behave exactly as a chain with $S = \{2, 3, 4, 5\}$ and transition matrix:

$$\begin{pmatrix} 0 & 1/4 & 1/4 & 1/2 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/4 & 1/2 & 0 & 1/4 \\ 1/4 & 1/4 & 1/2 & 0 \end{pmatrix}$$

This chain is regular so it has a limiting distribution $\begin{pmatrix} w_2 & w_3 & w_4 & w_5 \end{pmatrix}$. Now by our earlier comparison, the original chain has a limiting distribution and it is given by $\begin{pmatrix} 0 & w_2 & w_3 & w_4 & w_5 \end{pmatrix}$.

(c) Theorem 4.7 says that if a chain is regular then it has a limiting distribution. The existence of a chain which is not regular but does have a limiting distribution (such as this example) does not violate this Theorem. It just says that the converse implication does not hold.

(d) The limiting distribution is also the unique equilibrium distribution so to find it we solve

$$\begin{aligned} w_2 &= \tfrac{1}{2}w_3 + \tfrac{1}{4}w_4 + \tfrac{1}{4}w_5 \\ w_3 &= \tfrac{1}{4}w_2 + \tfrac{1}{2}w_4 + \tfrac{1}{4}w_5 \\ w_4 &= \tfrac{1}{4}w_2 + \tfrac{1}{4}w_3 + \tfrac{1}{2}w_5 \\ w_5 &= \tfrac{1}{2}w_2 + \tfrac{1}{4}w_3 + \tfrac{1}{4}w_4 \end{aligned}$$

Solving these[1] we get that the limiting distribution is $\begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$. So (by Theorem 4.9) the expcetation of the proportion of time spent in state 5 tends to $\frac{1}{4}$.

[1] If you spotted a quick way to solve these then well done. If you didn't can you see it now? Hint: Sheet 3, Question 3(d).

2. Looking at the matrix, some communicating classes should be obvious. After that a good way to spot the communicating classes is to draw the transition graph. We went through some of the thought processes for doing this in the Week 6 seminar. You get that:

- State 7 is absorbing so $\{7\}$ is a communicating class;
- There is no way to leave the set of states $\{8,9\}$ so this set is a communicating class;
- From states $\{3,4\}$ the only paths out lead to state 7 or state 9 and we can never retirn from these so this set is a communicating class;
- It is possible to move freely among the states $\{1,2,5,6\}$ (for instance by following the cycle $1 \to 6 \to 2 \to 5 \to 1$) so this is a communicating class.

So the communicating classes are:

$$\{1,2,5,6\}, \{3,4\}, \{7\}, \{8,9\}.$$

Now we need to classify them as recurrent or transient:

- State 7 is certainly recurrent since it is absorbing.
- States 8 and 9 are also recurrent since once we reach this class we can never leave it and so we must return (for instance the probability that we do not return to 8 within t steps or less is $\frac{6}{7}\left(\frac{1}{4}\right)^{t-1}$ which tends to 0).
- Every other state is transient. To see this we just need to show that the probability of never returning is positive. For state 1 the probability of going straight to state 7 is 1/2 and so the probability of never returning to 1 is at least 1/2 (so is certainly positive). Similarly,

$$\begin{aligned}
\mathbb{P}(\text{never return to } 2) &\geqslant p_{2,8} > 0\\
\mathbb{P}(\text{never return to } 3) &\geqslant p_{3,7} > 0\\
\mathbb{P}(\text{never return to } 4) &\geqslant p_{4,9} > 0\\
\mathbb{P}(\text{never return to } 5) &\geqslant p_{5,1}p_{1,7} > 0\\
\mathbb{P}(\text{never return to } 6) &\geqslant p_{6,2}p_{2,8} > 0
\end{aligned}$$

  You might have given a different argument which involves only having to check probabilities like this for two of the transient states rather than all of them. If you didn't do this can you see that argument now?

3.

(a) Clearly $f_1^{(1)} = 1/2$. From the transition diagram (or the matrix) we can see that to start at 1 and return there in more than one step we must go first to state 2, then go around the loop $2, 3, 4, 2$ some number of times (possibly 0), and finally return to 1. It follows that

- If $t > 1$ is not of the form $3r + 2$ for some integer $r \geqslant 0$ then $f_1^{(t)} = 0$.
- If $t = 3r + 2$ for some integer $r \geqslant 0$ then

$$f_1^{(t)} = p_{12} \left(p_{2,3} p_{3,4} p_{4,2}\right)^r p_{2,1} = \frac{1}{8} \left(\frac{2}{5}\right)^{\frac{t-2}{3}}.$$

So

$$f_1^{(t)} = \begin{cases} \frac{1}{2} & \text{if } t = 1 \\ \frac{1}{8}\left(\frac{2}{5}\right)^{\frac{t-2}{3}} & \text{if } t = 3r+2 \text{ for some } r \in \mathbb{N} \cdot \\ 0 & \text{otherwise.} \end{cases}$$

To find f_1 we have essentially a geometric progression (with a extra term at the start)

$$f_1 = \frac{1}{2} + \frac{1}{8} + \frac{1}{8}\left(\frac{2}{5}\right) + \frac{1}{8}\left(\frac{2}{5}\right)^2 + \cdots = \frac{1}{2} + \frac{5}{24} = \frac{17}{24}.$$

(b) State 1 is transient because $f_1 < 1$.

(c) If you only wanted to establish that state 1 is transient, the exact value of f_1 as calculated in part (a) is not needed. It is enough to note that

$$\mathbb{P}(\text{never return to } 1) \geqslant p_{1,2} p_{2,3} p_{3,5} = \tfrac{1}{2}\tfrac{3}{4}\tfrac{1}{5} = \tfrac{3}{40}$$

so $f_1 \leqslant \frac{37}{40} < 1$.

(d) If state 2 were modified in this way then there would be more possible paths giving a first return to 1 at time t. To calculate $f_1^{(t)}$ we would have to work out all such paths and sum over them which would be fiddly.

(e) The method of first step analysis can be used to do this. Suppose that after the first step we make state 1 into an absorbing state then the probability of return f_1 in the original Markov chain is the probability that in this new chain we are absorbed at 1. This can be calculated by first step analysis.

More explicitly if we define $a_i = \mathbb{P}(X_t = 1 \text{ for some } t > 0 \mid X_0 = i)$ for $i \neq 1$ then the a_i satisfy $a_i = p_{i1} + \sum_{s \in S \setminus \{1\}} p_{is} a_s$ (with $a_5 = 0$ of course). These equations allow us to calculate the a_i. Finally, we have (again by conditioning on the first step) that $f_1 = \frac{1}{2} + \frac{1}{2} a_2$. Try it and check you get the same answer as in part (a).

4. The thing we are trying to prove relates the first return probabilities $f_i^{(t)}$ and the t-step transition probabilities $p_{ii}^{(t)}$. Let's think how we could make an expression which includes both of these. One idea would be to calculate $p_{ii}^{(t)}$ by conditioning on the time of first return:

$$
\begin{aligned}
p_{ii}^{(t)} &= \sum_{k=1}^{t} \mathbb{P}(X_t = i \mid \text{first return to } i \text{ is at time } k)\mathbb{P}(\text{first return to } i \text{ is at time } k) \\
&= \sum_{k=1}^{t} \mathbb{P}(X_t = i \mid X_k = i, X_{k+1} \neq i, \ldots, X_1 \neq i, X_0 = i)\mathbb{P}(X_k = i, X_{k+1} \neq i, \ldots, X_1 \neq i \mid X_0 = i) \\
&= \sum_{k=1}^{t} \mathbb{P}(X_t = i \mid X_k = i) f_i^{(k)} \\
&\qquad \text{(by the Markov property and the definition of } f_i^{(k)}\text{)} \\
&= \sum_{k=1}^{t} p_{ii}^{(t-k)} f_i^{(k)} \\
&= p_{ii}^{(0)} f_i^{(t)} + \sum_{k=1}^{t-1} p_{ii}^{(t-k)} f_i^{(k)} \qquad \text{(splitting off the } k = t \text{ term)} \\
&= f_i^{(t)} + \sum_{k=1}^{t-1} p_{ii}^{(t-k)} f_i^{(k)} \qquad \text{(since } p_{ii}^{(0)} = 1\text{)}
\end{aligned}
$$

Rearranging this gives us the expression we wanted.

Please let me know if you have any comments or corrections

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