Random Processes – 2023/24 Problem Sheet 5

The questions on this sheet are based on the material from from Week 5 lectures. Question 1 is a fairly standard example on equilibrium and limiting distribution but with a small twist. Question 2 is about communicating classes and the general idea of recurrent and transient states. Questions 3 and 4 are about first-return probabilities.

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1. Let $(X_0, X_1, \ldots)$ be the Markov chain with state space $S = \{1, 2, 3, 4, 5\}$ and transition matrix:

$$\begin{pmatrix} 99/100 & 1/100 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/2 \\ 0 & 1/2 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 1/2 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 1/2 & 0 \end{pmatrix}$$

(a) Is this chain regular?

(b) Does it have a limiting distribution?

(c) Explain why your answers to parts (a) and (b) do not violate Theorem 4.7.

(d) What can you say about the proportion of time spent in state 5 in the long run?

2. Consider the Markov chain on state space $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ with transition matrix

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 1/6 & 0 & 0 & 1/6 & 0 \\ 0 & 0 & 1/4 & 1/4 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 9/10 & 0 & 0 & 0 & 0 & 0 & 1/10 \\ 1/2 & 1/3 & 1/6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/7 & 6/7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/4 & 1/4 \end{pmatrix}.$$

Find the communicating classes of states of the chain. For each state say whether it is recurrent or transient, giving a brief explanation of why (you do not need to calculate the first return probabilities exactly to do this).

3. Consider the Markov chain on state space $\{1,2,3,4,5\}$ with transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 4/5 & 1/5 \\ 0 & 2/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(a) Calculate the first return probability $f_1^{(t)}$ and the return probability f_1.

(b) Is state 1 recurrent or transient?

(c) Did you need your solution to part (a) to answer part (b)?

(d) Why would it become harder to calculate $f_1^{(t)}$ if state 2 was modified so that p_{21}, p_{22}, p_{23} were all positive?

(e) How could you calculate f_1 without working out the $f_1^{(t)}$?

4. Prove that for any Markov chain, any $i \in S$ and any $t \geqslant 1$ we have

$$f_i^{(t)} = p_{ii}^{(t)} - \sum_{k=1}^{t-1} p_{ii}^{(t-k)} f_i^{(k)}.$$

[Hint: Work out $p_{ii}^{(t)}$ by conditioning on the time of the first return to i.]

5. [Challenge Question]

(a) Prove that $\leftrightarrow$ (intercommunicates) is an equivalence relation on S.

(b) Show that every Markov chain on a finite S has at least one communicating class which is recurrent. Is the same result true when S is infinite?

Some recent exam questions on the material in Week 5 include:

- Main Exam Period 2018. Question 5 (a,b) (The notation $f_{i,i}$ is used instead of f_i for the return probability here.)
- January 2022 Exam. Question 2 (b,c,d)
- January 2023 Exam. Question 1 (e,f)

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