1.

(a) From the transition graph this is clearly irreducible. Again looking at the transition graph (or powers of the matrix) it is possible to get between any two states in 3 steps. So the chain is regular.

Since it is regular it has a limiting distribution (by Theorem 4.7) and that limiting distribution is the unique limiting distribution (by Theorem 4.6).

In Problem Sheet 3 we found this equilibrium distribution was

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} = \begin{pmatrix} 24/73 & 18/73 & 16/73 & 15/73 \end{pmatrix}$$

and so this is the limiting distribution.

(b) This chain is not irreducible (there is no way of getting from state 2 to state 1). Since it is not irreducible it is certainly not regular.

So Theorem 4.7 tells us nothing and we need to think.

States 2 and 3 are both absorbing so:

$$p_{2,2}^{(r)} = \mathbb{P}(X_r = 2 \mid X_0 = 2) = 1$$
$$p_{3,2}^{(r)} = \mathbb{P}(X_r = 2 \mid X_0 = 3) = 0$$

So P^r cannot tend to a limit with all rows equal (equivalently $\mathbb{P}(X_r = 2)$ does not tend to a limit which does not depend on X_0). So there is no limiting distribution.

(c) This chain is not irreducible (there is no way of getting from state 3 to state 1). Since it is not irreducible it is certainly not regular.

Again Theorem 4.7 tells us nothing and we need to think.

The only way we can fail to reach state 3 is for us to loop at state 1. The probability that this happens up to time r is $\left(\frac{2}{5}\right)^r \to 0$ as $r \to \infty$. So we are absorbed at 3 with probability 1. So there is a limiting distribution and it is

$$\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

Equivalently we have

$$P^r \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \text{ as } r \to \infty.$$

(d) This chain is regular since P has no 0 entries. So it is certainly irreducible. So (by Theorem 4.7) it has a limiting distribution. We checked in Problem Sheet 3 that the vector $\mathbf{w} = \begin{pmatrix}1/4 & 1/4 & 1/4 & 1/4\end{pmatrix}$ is an equilibrium distribution. By Theorem 4.6 this is also the limiting distribution.

(e) Again, this chain is regular since P has no 0 entries. So it is certainly irreducible.

We don't need any theory to see it has a limiting distribution. The structure of the matrix means that $P^r = P$ for all r. Hence $P^r \to P$ as $r \to \infty$. The matrix P has all rows equal to $\begin{pmatrix}a & b & c & d\end{pmatrix}$ and so this is the limiting distribution.

2.

(a) If $p = 1$ then state 1 is an absorbing state so the chain is not irreducible (for instance $p_{1,2}^{(r)} = 0$ for all r).

If $p \neq 1$ then the chain is irreducible. This is fairly easy to see from the transition graph. If you have trouble seeing this notice that there is a cycle starting in state 1 and following arrows corresponding to transitions: $1 \to 5 \to 3 \to 2 \to 5 \to 4 \to 1$. Starting from any state we can follow this cycle to get to any other state.

(b) If $p = 1$ then it is certainly not regular because being regular is a stronger condition than irreducible. If $0 < p < 1$ then it is regular by Question 3(a): the chain is irreducible and has a loop at state 1. That leaves $p = 0$ as the tricky case

If $p = 0$ then you can see from the transition graph that there the chain has a cyclic structure.

- From states 1 and 2 we must move to state 5.
- From state 5 we must move to state 3 or 4.
- From states 3 and 4 we must move to state 1 or 2.

From this you can see that the chain is not regular. For instance: if r is a multiple of 3 then $p_{1,5}^{(r)} = 0$ while if r is not a multiple of 3 then $p_{1,1}^{(r)} = 0$. So there is no r for which $p_{i,j}^{(r)} \neq 0$ for all $i, j \in \{1, 2, 3, 4, 5\}$.

(c) Since the chain is regular it has a limiting distribution $\mathbf{w}$ (by Theorem 4.7). Moreover (by Theorem 4.6) $\mathbf{w}$ is the unique solution to $\mathbf{w}P = \mathbf{w}$. We therefore solve

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \\ 1/4 & 3/4 & 0 & 0 & 0 \\ 3/4 & 1/4 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & w_5 \end{pmatrix}$$

That is

$$\begin{aligned} w_1 &= \tfrac{1}{2}w_1 + \tfrac{1}{4}w_3 + \tfrac{3}{4}w_4 \\ w_2 &= \tfrac{3}{4}w_3 + \tfrac{1}{4}w_4 \\ w_3 &= \tfrac{1}{2}w_5 \\ w_4 &= \tfrac{1}{2}w_5 \\ w_5 &= \tfrac{1}{2}w_1 + w_2 \end{aligned}$$

The third and fourth equations give that $w_3 = w_4 = \frac{1}{2}w_5$. So the first two equations become:

$$\begin{aligned} w_1 &= \tfrac{1}{2}w_1 + w_3 \quad \text{so} \quad w_1 = 2w_3 \\ w_2 &= w_3 \end{aligned}$$

Finally the last equation gives

$$w_5 = 2w_3$$

Putting this all together we get that

$$\mathbf{w} = \begin{pmatrix} 2w_3 & w_3 & w_3 & w_3 & 2w_3 \end{pmatrix}$$

Since $\mathbf{w}$ is a probability vector we have $7w_3 = 1$ so $w_3 = \frac{1}{7}$ and

$$\mathbf{w} = \begin{pmatrix} \frac{2}{7} & \frac{1}{7} & \frac{1}{7} & \frac{1}{7} & \frac{2}{7} \end{pmatrix}$$

It is a good idea after a calculation like this to check by doing the matrix multiplication $\mathbf{w}P$ and making sure that you do get $\mathbf{w}$.

(d) After the robot has been running for a long time, the chance that it will be in each region will be very close to the numbers in part (c). For instance the chance that it is in region 1 at a particular large time will be very close to $\frac{2}{7}$ regardless of where it started.

3.

(a) Since the chain is irreducible we know that for any $a, b \in S$ there is an r with $p_{ab}^{(r)} > 0$. Let m be the maximum of all of these rs (we can do this since there are only finitely many pairs of states). We will show that for any two states a, b we have $p_{ab}^{(2m)} > 0$.

It follows from the definition of m that given any pair of states $a, b \in S$ we have that $p_{ai}^{(s)} > 0$ for some $s \leqslant m$ and $p_{ib}^{(t)} > 0$ for some $t \leqslant m$. So there is a path from a to b via i of length $s + t \leqslant 2m$ steps. Since $p_{ii} > 0$ we can add loops at state i to make this into a path from a to b with $2m$ steps. So the chain is regular (setting $r = 2m$ in the definition will work).

To see this a bit more formally,

$$p_{ab}^{(2m)} \geqslant p_{ai}^{(s)} p_{ii}^{(2m-s-t)} p_{ib}^{(t)} > 0$$

because $2m - s - t \geqslant 0$ and $p_{ii}^{(2m-s-t)} \geqslant p_{ii}^{2m-s-t}$.

(b) The result is no longer true for infinite S. Consider the Markov chain with $S = \mathbb{Z}$ and

$$p_{i,i} = p_{i,i+1} = p_{i,i-1} = 1/3 \quad \text{for all } i \in \mathbb{Z},$$

and all other transition probabilities 0. This is irreducible but given any r we have that $p_{0,r+1}^{(r)} = 0$ (we cannot get from state 0 to state $r+1$ in only r steps). and so it is not regular. (The part of the proof above which fails here is that we cannot necessarily take the maximum of an infinite set of numbers).

4.

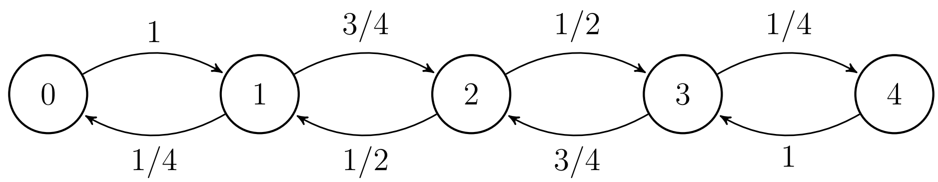
(a) We know that the total number of balls is always n. So we can describe the state of the system by saying how many balls are in the first bucket. This gives $n+1$ states where state k means 'there are k balls in the first bucket and $n-k$ balls in the second bucket'.

The transitions involve moving a ball from one bucket to another so the state number only changes by 1. We have

$$p_{i,i-1} = \mathbb{P}(X_{t+1} = i-1 \mid X_t = i) = \mathbb{P}(\text{ball chosen from first bucket}) = \frac{i}{n} \qquad \text{(for all } 1 \leqslant i \leqslant n)$$

$$p_{i,i+1} = \mathbb{P}(X_{t+1} = i+1 \mid X_t = i) = \mathbb{P}(\text{ball chosen from second bucket}) = \frac{n-i}{n} \qquad \text{(for all } 0 \leqslant i \leqslant n-1)$$

(b) When $n = 4$ the transition graph is



and the transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

(c) The chain is irreducible (we can get from any state to any other state) However, every step involves moving from an odd state to an even state. This means that the powers of P must have the following shape:

$$P^r = \begin{pmatrix} 0 & ? & 0 & ? & 0 \\ ? & 0 & ? & 0 & ? \\ 0 & ? & 0 & ? & 0 \\ ? & 0 & ? & 0 & ? \\ 0 & ? & 0 & ? & 0 \end{pmatrix} \text{ when } r \text{ is odd,} \quad P^r = \begin{pmatrix} ? & 0 & ? & 0 & ? \\ 0 & ? & 0 & ? & 0 \\ ? & 0 & ? & 0 & ? \\ 0 & ? & 0 & ? & 0 \\ ? & 0 & ? & 0 & ? \end{pmatrix} \text{ when } r \text{ is even.}$$

From this we can see that the chain is not regular (there are 0 entries in P^r for every r). We can also see that there is no limiting distribution because the only way such a sequence could tend to a limit would be if the limit was the all 0 matrix and the rows of this are not probability vectors (in fact P^r does not tend to a limit).

(d) The chain is irreducible so by Theorem 4.8 it has a unique equilibrium distribution.

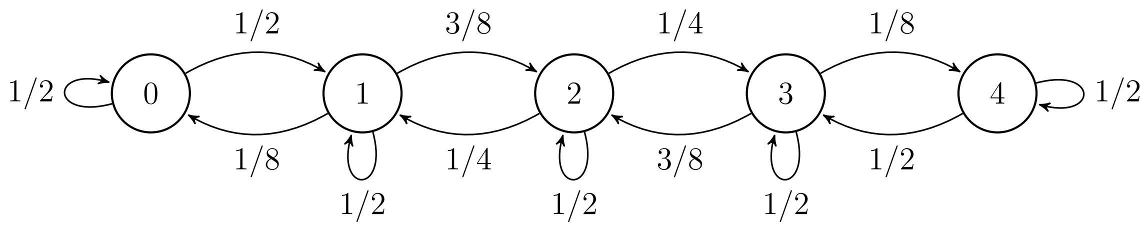
(e) To find the equilibrium distribution when $n = 4$ we solve $\mathbf{w}P = \mathbf{w}$. That is

$$\begin{pmatrix} w_0 & w_1 & w_2 & w_3 & w_4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} w_0 & w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

subject to $w_0 + w_1 + w_2 + w_3 + w_4 = 1$. You get (I leave the details to you):

$$\mathbf{w} = \begin{pmatrix} 1/16 & 1/4 & 3/8 & 1/4 & 1/16 \end{pmatrix}$$

(f) Suppose we put the ball back in a random bucket rather than moving it. We have a 1/2 chance of putting it back in the same bucket (in which case the state doesn't change) and a 1/2 chance of moving it (in which case we make the same move as in the original chain). So in the transition graph we add a loop labelled 1/2 at each state and scale all other probabilities by 1/2. For instance when $n = 4$ we get



Equivalently, the transition matrix for the new chain is $\frac{1}{2}P + \frac{1}{2}I_n$ (where I_n is the $n \times n$ identity matrix).

The chain is now regular (see Question 3a for instance) so it has a limiting distribution which is also the unique limiting distribution. This equilibrium distribution will be the same as the equilibrium distribution for the original chain (because $\mathbf{w}P = \mathbf{w}$ if and only if $\mathbf{w}\left(\frac{1}{2}P + \frac{1}{2}I_n\right) = \mathbf{w}$).

Please let me know if you have any comments or corrections

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