2.

(a) The eigenvalues of P are 1 and 1/4 with eigenvectors $\binom{1}{1}$ and $\binom{1}{-1/2}$. It follows that:

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$$

(Look back at your Linear Algebra I notes if you need a reminder of diagonalisation.)

So

$$P^5 = \begin{pmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1024} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} + \frac{1}{1536} & \frac{2}{3} - \frac{1}{1536} \\ \frac{1}{3} - \frac{1}{3072} & \frac{2}{3} + \frac{1}{3072} \end{pmatrix}$$

(b) The matrix P^5 gives the 5-step transition probabilities so we can read off the matrix that

$$\begin{aligned} \mathbb{P}(X_5 = 1 \mid X_0 = 1) = p_{11}^{(5)} &= \frac{1}{3} + \frac{1}{1536} \\ \mathbb{P}(X_5 = 1 \mid X_0 = 2) = p_{21}^{(5)} &= \frac{1}{3} - \frac{1}{3072} \end{aligned}$$

(c) Similarly to part (a) we have that

$$P^{100} = \begin{pmatrix} 1 & 1 \\ 1 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{4^{100}} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} + \frac{2}{3\times 4^{100}} & \frac{2}{3} - \frac{2}{3\times 4^{100}} \\ \frac{1}{3} - \frac{1}{3\times 4^{100}} & \frac{2}{3} + \frac{1}{3\times 4^{100}} \end{pmatrix}$$

and so

$$\begin{aligned} \mathbb{P}(X_{100} = 1 \mid X_0 = 1) &= \frac{1}{3} + \frac{2}{3 \times 4^{100}} \\ \mathbb{P}(X_{100} = 1 \mid X_0 = 2) &= \frac{1}{3} - \frac{1}{3 \times 4^{100}} \end{aligned}$$

(d) The probabilities in part (b) are very close and those in part (c) are even closer. This is saying that the process forgets its starting state in the sense that $\mathbb{P}(X_{100} = 1 \mid X_0 = i)$ does not depend very much on i. If you were to calculate $\mathbb{P}(X_n = 1 \mid X_0 = 1)$ and $\mathbb{P}(X_n = 1| \mid X_0 = 2)$ for even larger n they would be even closer and in the limit as $n \to \infty$ they are equal.

3.

(a) To find the equilibrium distributions we need to solve:

$$\begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/4 & 3/4 \\ 4/5 & 0 & 0 & 1/5 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

subject to $w_1 + w_2 + w_3 + w_4 = 1$. You get (I leave the details to you) $\begin{pmatrix} w_1 & w_2 & w_3 & w_4 \end{pmatrix} = \begin{pmatrix} 24/73 & 18/73 & 16/73 & 15/73 \end{pmatrix}$ as the unique equilbrium distribution.

(b) Considering

$$\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} 2/5 & 1/5 & 2/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix}$$

we get the equations

$$\begin{aligned} \tfrac{2}{5}w_1 &= w_1 \\ \tfrac{1}{5}w_1 + w_2 &= w_2 \\ \tfrac{2}{5}w_1 + w_3 &= w_3 \end{aligned}$$

From the first equation we must have $w_1 = 0$ and then any choice of w_2, w_3 will satisfy the second and third equations.

So any probability vector $\mathbf{w} = \begin{pmatrix} 0 & \alpha & 1-\alpha \end{pmatrix}$ is an equilibrium distribution.

(c) The only probability vector which is a solution to $\mathbf{w}P = \mathbf{w}$ is $\mathbf{w} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$ so this is the unique equilibrium distribution.

(d) By inspection the vector $\mathbf{w} = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$ satisfies $\mathbf{w}P = \mathbf{w}$ and so this is an equilibrium distribution. It turns out that this is the unique equilibrium distribution. This follows from some theory we see in Week 4.

(e) The first entry of the usual matrix equations gives

$$aw_1 + aw_2 + aw_3 + aw_4 = w_1$$

Since $w_1 + w_2 + w_3 + w_4 = 1$ this gives $w_1 = a$. Similarly $w_2 = b$, $w_3 = c$, $w_4 = d$ so the unique equilibrium distribution is $\begin{pmatrix} a & b & c & d \end{pmatrix}$.

This Markov chain is not very interesting. The fact that the rows of P are all equal means that for every t we have $\mathbb{P}(X_t = 1) = a$, $\mathbb{P}(X_t = 2) = b$, $\mathbb{P}(X_t = 3) = c$, $\mathbb{P}(X_t = 4) = d$ and these do not depend on X_{t-1}. In other words, $(X_0, X_1, \ldots)$ is just a sequence of mutually independent random variables.

4.

(a) This is impossible. Suppose that $\mathbf{w}$ and $\mathbf{u}$ are both equilibrium distribiutions. I claim that $\mathbf{x} = \frac{1}{2}\mathbf{w} + \frac{1}{2}\mathbf{u}$ is also an equilibrium distribution. It is easy to see that $\mathbf{x} \neq \mathbf{u}$, and $\mathbf{x} \neq \mathbf{w}$. So we cannot have exactly two equilibrium distributions because from any two we can construct a third.

Now $\mathbf{x}$ is certainly a probabilty vector so to check it is an equilibirum distribution we need to check that $\mathbf{x}P = \mathbf{x}$.

$$\begin{aligned}
\mathbf{x}P &= \left(\tfrac{1}{2}\mathbf{w} + \tfrac{1}{2}\mathbf{u}\right) P \\
&= \tfrac{1}{2}\mathbf{w}P + \tfrac{1}{2}\mathbf{u}P \\
&= \tfrac{1}{2}\mathbf{w} + \tfrac{1}{2}\mathbf{u} && \text{(since } \mathbf{w} \text{ and } \mathbf{u} \text{ are equilibrium distributions)} \\
&= \mathbf{x}.
\end{aligned}$$

This completes the proof.

In fact, for any $0 < \alpha < 1$ we have that $\alpha\mathbf{w} + (1-\alpha)\mathbf{u}$ is an equilibrium distribution. So we either have a unique equilibrium distribution or infinitely many of them.

(b) This is possible. Indeed, the Markov chain in Question 3(c) above is an example. This has one absorbing state (state 3) and a unique equilibrium distribution.

(c) If state k is absorbing then the probability vector

$$w_i = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

is an equilibirum distribution (just check the matrix equation $\mathbf{w}P = \mathbf{w}$ is satisfied). If there are three absorbing states we can construct three different equilibrium distributions so it is not possible for there to be a unique equilibrium distribution.

5. Let d_i be the number of passages coming out of room i. The vector $\mathbf{u} = \begin{pmatrix} d_1 & d_2 & \cdots & d_n \end{pmatrix}$ satisfies $\mathbf{u}P = \mathbf{u}$. Showing this is equivalent to showing:

$$\sum_{s=1}^{n} p_{si} d_s = d_i \quad \text{for all } i.$$

Now $p_{si} = 1/d_s$ if there is a passage from s to i and is 0 otherwise. It follows that

$$\sum_{s=1}^{n} p_{si} d_s = 1 \times (\text{the number of passages out of } i) = d_i.$$

If we normalise this vector by letting $d = d_1 + d_2 + \cdots + d_n$ and $\mathbf{w} = \frac{1}{d}\mathbf{u}$ then $\mathbf{w}$ is an equilibrium distribution.

Please let me know if you have any comments or corrections

Robert Johnson
`r.johnson@qmul.ac.uk`