Random Processes – 2023/24 **Problem Sheet 3**

The questions on this sheet are based on the material on long-term behaviour from Week 3 lectures. Question 1 asks you to reflect on what we have seen so far. Question 2 is a concrete example of calculating multi-step transition probabilities by diagonalising the transition matrix. Questions 3 (more computational) and 4 (more theoretical) both explore the idea of equilibrium distributions. We will discuss selected parts in the Week 4 seminars.
I would like you to submit your answer to **Question 1** by **5pm on Friday 20 October 2023** via the QMplus page following the instructions there. These will not be marked but I will give some general feedback and I am interested to see what you pick out.

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1. Which result or concept from the first three weeks of the module appeals to you the most? Say what it is about this that you particularly like and why.

2. Let $X_0, X_1, \dots$ be the Markov chain on state space $\{1, 2\}$ with transition matrix

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}.$$

(a) By diagonalising P find P^5.

(b) Find $\mathbb{P}(X_5 = 1 \mid X_0 = 1)$ and $\mathbb{P}(X_5 = 1 \mid X_0 = 2)$.

(c) Find $\mathbb{P}(X_{100} = 1 \mid X_0 = 1)$ and $\mathbb{P}(X_{100} = 1 \mid X_0 = 2)$.

(d) Comment on your answers to (b) and (c).

3. For each of the following transition matrices find all equilibrium distributions for the corresponding Markov chains.

(a)

$$\begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/4 & 3/4 \\ 4/5 & 0 & 0 & 1/5 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 2/5 & 1/5 & 2/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 2/5 & 1/5 & 2/5 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

(d)

$$\begin{pmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}$$

Where $a+b+c+d=1$ and $0<a,b,c,d<1$.

(e)

$$\begin{pmatrix} a & b & c & d \\ a & b & c & d \\ a & b & c & d \\ a & b & c & d \end{pmatrix}$$

Where $a+b+c+d=1$ and $0<a,b,c,d<1$. What else can you say about this Markov chain?

4. For each of the following, decide, whether it is possible for a Markov chain with this property to exist. Justify your answer appropriately.

(a) There are exactly two equilibrium distributions.

(b) There is a unique equilibrium distribution and at least one absorbing state.

(c) There a unique equilibrium distribution and exactly 3 absorbing states.

5. [Challenge Question] Recall our rooms and passages example from lecture 1. We take a (finite) set S of rooms with passages between some pairs of rooms. Suppose also that we do not have any loops (passages from a room to itself). Define d_i to be the number of passages coming out of room i.

At each time step we choose a passage leading from our current location at random (with each such choice equally likely) and move along it. Setting X_t to be our position (room number) after t time steps gives a Markov chain

(a) Show that for any such configuration of rooms there is an equilibrium distribution, and specify it in terms of the d_i.

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