1.

(a) States 1 and 4 are absorbing (because $p_{11} = p_{44} = 1$).

(b) To do this we use first step analysis. Let T be the time of absorption. Define

$$a_i = \mathbb{P}(X_T = 1 \mid X_0 = i).$$

The question asks us to find a_2.

Plainly $a_1 = 1$ and $a_4 = 0$. Now using first-step analysis (that is conditioning on the first step the chain takes) we obtain the equations:

$$\begin{aligned} a_2 &= \tfrac{1}{4} + \tfrac{1}{4}a_2 + \tfrac{1}{4}a_3 \\ a_3 &= \tfrac{1}{6} + \tfrac{1}{6}a_2 + \tfrac{1}{6}a_3. \end{aligned}$$

Solving these gives $\mathbb{P}(X_T = 1) = a_2 = \frac{3}{7}$.

2.

(a) States 4 and 5 are absorbing (because $p_{44} = p_{55} = 1$).

(b) Let T be the time of absorption and define $a_i = \mathbb{P}(X_T = 4|X_0 = i)$. By first-step analysis we have

$$\begin{aligned} a_1 &= a_3 \\ a_2 &= \tfrac{4}{5}a_3 + \tfrac{1}{5} \\ a_3 &= \tfrac{1}{6}a_2 + \tfrac{2}{3}a_3. \end{aligned}$$

Solving these gives $\mathbb{P}(X_T = 4) = a_1 = \frac{1}{6}$.

(c) Let $u_i = \mathbb{E}(T|X_0 = i)$. By first step analysis (using Corollary 3.3 from the notes) we have

$$\begin{aligned} u_1 &= 1 + u_3 \\ u_2 &= 1 + \tfrac{4}{5}u_3 \\ u_3 &= 1 + \tfrac{1}{6}u_2 + \tfrac{2}{3}u_3. \end{aligned}$$

Solving these gives $\mathbb{E}(T) = u_1 = \frac{41}{6}$.

(d) Let $V = |\{n : 0 \leqslant n \leqslant T-1, X_n = 2\}|$. Let $v_i = \mathbb{E}(V \mid X_0 = i)$ be the expected number of visits to state 2 before absorption starting from state i. By first-step analysis (using Theorem 3.2 from the notes with $w(1) = w(3) = 0$, $w(2) = 1$) we have

$$\begin{aligned} v_1 &= v_3 \\ v_2 &= 1 + \tfrac{4}{5} v_3 \\ v_3 &= \tfrac{1}{6} v_2 + \tfrac{2}{3} v_3. \end{aligned}$$

Solving these gives $\mathbb{E}(V) = v_1 = 5/6$

(e) Let $w(1) = 0, w(2) = -5$ and $w(3) = 10$. Your gain is

$$G = \sum_{i=0}^{T-1} w(X_i).$$

Let $g_i = \mathbb{E}(G \mid X_0 = i)$ be your expected gain up to absorption starting from state i. By first-step analysis (using Theorem 3.2 from the notes)

$$\begin{aligned} g_1 &= g_3 \\ g_2 &= -5 + \tfrac{4}{5} g_3 \\ g_3 &= 10 + \tfrac{1}{6} g_2 + \tfrac{2}{3} g_3. \end{aligned}$$

Solving these gives $\mathbb{E}(G) = g_1 = \frac{275}{6}$.

(f) The process spends exactly $T - 1$ steps in states 2 and 3 ($X_0 = 1$ and $X_T = 4$ or 5, every other step must be in state 2 or 3). We know that V of them are in state 2 so

$$G = -5V + 10(T - 1 - V)$$

By linearity of expectation

$$\mathbb{E}(G) = -5\mathbb{E}(V) + 10\mathbb{E}(T) - 10 - 10\mathbb{E}(V) = -15\mathbb{E}(V) + 10\mathbb{E}(T) - 10$$

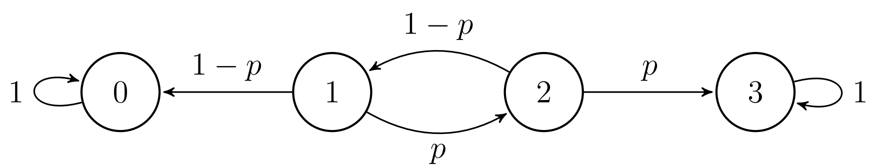
Putting in the numbers from parts (c) and (d) we get $\mathbb{E}(G) = 275/6$ as before.

3.

(a) The state of the game can be represented by stating the amount of money that we each have. However, if I have £k then you must have £$(3-k)$ (since the total amount of money is £3 throughout the game). So let the state space be $S = \{0, 1, 2, 3\}$ where state k represents "I have £k you have £$(3-k)$". We are told that the game stops when we reach state 0 or 3. We will represent this by making these two states absorbing. The transition probabilities are:

$$p_{00} = p_{33} = 1, \quad p_{12} = p_{23} = p, \quad p_{10} = p_{21} = 1 - p$$

with all other transitions having probability 0.



The Markov chain with these transition probabilities and $X_0 = 1$ describes the game.

(b) Let T be the time of absorption. The event I end up with no money is equivalent to the event $X_T = 0$. The event you end up with no money is equivalent to the event $X_T = 3$. Let

$$a_i = \mathbb{P}(X_T = 0 \mid X_0 = i),$$

and

$$b_i = \mathbb{P}(X_T = 3 \mid X_0 = i).$$

Our task is to find the p for which $a_1 = b_1$.

First, let's calculate a_1 and b_1 using first step analysis:

$$\begin{aligned} a_1 &= (1-p) + pa_2 \\ a_2 &= (1-p)a_1. \end{aligned}$$

Solving these gives $a_1 = \frac{1-p}{p^2-p+1}$. Also,

$$\begin{aligned} b_1 &= pb_2 \\ b_2 &= p + (1-p)b_1. \end{aligned}$$

Solving these gives $b_1 = \frac{p^2}{p^2-p+1}$. (You could also have worked out b_1 by noting that since we are certainly absorbed eventually we must have $b_1 = 1 - a_1$).

Hence if $a_1 = b_1$ we have that.

$$p^2 + p - 1 = 0.$$

This equation has solutions

$$p = \frac{-1 \pm \sqrt{5}}{2}.$$

Since p is a probability we take the positive solution

$$p = \frac{-1 + \sqrt{5}}{2}.$$

4.

(a) For all ordered pairs ij with $1 \leqslant i, j \leqslant 6$ take state ij to mean that the last roll was j and the previous roll was i. We set states $13, 22, 31$ to be absorbing since if we reach one of these the process ends. For all other states ab we set

$$\mathbb{P}(X_{t+1} = cd | X_t = ab) = \begin{cases} 1/6 & \text{if } b = c \\ 0 & \text{otherwise} \end{cases}$$

The first two rolls are equally likely to be any pair so we could let the chain start at time 2 with the starting state chosen uniformly at random. In other words, the process if $(X_2, X_3, \ldots)$ and $\mathbb{P}(X_2 = ab)\frac{1}{36}$ for all $ab \in S$. There are other ways to model the start but this is probably the neatest.

(b) We need to keep track of whether the process has finished so let's take one state representing this. If the process hasn't finished we only need to know whether the previous roll was $1, 2, 3$ or something else to decide whether our next roll ends the process. So take states as follows:

State 4: The last two rolls sum to 4 so the process ends (an absorbing state);

State 0: the last roll was 4,5 or 6;

States 1,2,3: state i (with $i = 1, 2, 3$) meaning that the last roll was i but the sum of the last two rolls is not 4.

Suppose that the last roll was a 1 then:

- the process will end if the next roll is a 3 so $p_{14} = 1/6$;
- if we roll 1 we move to state 1 so $p_{11} = 1/6$;
- if we roll 2 we move to state 2 so $p_{12} = 1/6$;
- if we roll 4, 5 or 6 we move to state 0 so $p_{30} = 1/2$.

You can work out the transition probabilities for other states similarly. We get that the transition matrix is

$$\begin{pmatrix} 1/2 & 1/6 & 1/6 & 1/6 & 0 \\ 1/2 & 1/6 & 1/6 & 0 & 1/6 \\ 1/2 & 1/6 & 0 & 1/6 & 1/6 \\ 1/2 & 0 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(where the five rows of the matrix are indexed by $0, 1, 2, 3, 4$).

(c) If we set $X_0 = 0$ then the number of rolls the process lasts is equal to the time of absorption.

Let $u_i = \mathbb{E}(T|X_0 = i)$. By first-step analysis we have

$$\begin{aligned} u_0 &= 1 + \tfrac{1}{6}u_1 + \tfrac{1}{6}u_2 + \tfrac{1}{6}u_3 + \tfrac{1}{2}u_0 \\ u_1 &= 1 + \tfrac{1}{6}u_1 + \tfrac{1}{6}u_2 + \tfrac{1}{2}u_0 \\ u_2 &= 1 + \tfrac{1}{6}u_1 + \tfrac{1}{6}u_3 + \tfrac{1}{2}u_0 \\ u_3 &= 1 + \tfrac{1}{6}u_2 + \tfrac{1}{6}u_3 + \tfrac{1}{2}u_0 \end{aligned}$$

Solving these gives $\mathbb{E}(T) = u_0 = 14$.

In fact there is a way of doing this with an even smaller number of states which you might have found. Hint: Is there any difference in behaviour between states $1, 2$ and 3?

5. The simplest example is probably to take two absorbing states and a pair of states which are not absorbing which the chain can never leave.
For instance take $S = \{1, 2, 3, 4, 5\}, X_0 = 1$ and transition matrix

$$P = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{pmatrix}$$

Notice that this doesn't satisfy the conditions of part (ii) of Theorem 3.1 from the notes; there are states (4 and 5) from which we can never reach an absorbing state. The chain is either absorbed immediately (with probability 2/3), or goes to state 4 in which case it is never absorbed (with probability 1/3).
The first step analysis equations for $a_i = \mathbb{P}(X_T = 2|X_0 = i)$ are $a_2 = 1, a_3 = 0$ and

$$\begin{aligned} a_1 &= \frac{1}{3} + \frac{1}{3}a_4 \\ a_4 &= \frac{1}{2}a_4 + \frac{1}{2}a_5 \\ a_5 &= \frac{1}{2}a_4 + \frac{1}{2}a_5 \end{aligned}$$

These do not have a unique solution; we could set $a_4 = a_5$ to be anything. Of course in this example it is impossible to get from state 4 or state 5 to state 1 so we have $a_4 = a_5 = 0$ and hence $a_1 = 1/3$. That was easy to do, but it involved a small argument about the chain; we can't just read the answer off the first-step analysis equations.
You will get similar bahaviour to this whatever example you found.

Please let me know if you have any comments or corrections

Robert Johnson
r.johnson@qmul.ac.uk