Complex Networks (MTH6142)
Solutions of Formative Assignment 5

- **1. Random networks in the $\mathbb{G}(N,p)$ ensemble**

  Assume that $p = a/N^z$, where $a > 0$ and $z \geq 0$, and a, z independent of N.

  (a) Determine the average degree $\langle k \rangle$ in the limit $N \to \infty$ for the following values of the parameters

  (i) $a = 0.5, z = 1$;
  (ii) $a = 2, z = 1$;
  (iii) $a > 0,\ z = 2$;
  (iv) $a > 0, z = 0.5$.

  (b) In which of the above cases does the random network contain a giant component in the limit $N \to \infty$?.

  (c) Given $p = a/N^z$ with generic values of $a > 0, z \geq 0$ determine the average degree $\langle k \rangle$ in the large network limit $N \to \infty$.

  (d) Determine the conditions on a and z for these random networks to be subcritical, i.e. with a fraction S of nodes in the giant component given by $S = 0$ in the $N \to \infty$ limit.

  (e) Determine the conditions on a and z for these random networks to be supercritical, i.e. with a non vanishing fraction S of nodes in the giant component $(S > 0)$ in the $N \to \infty$ limit.

  (f) Determine the conditions on a and z for which these random networks are critical, in the large network limit, i.e. in the limit $N \to \infty$.

- *Notes on the solution*

(a)& (b) The average degree $\langle k \rangle$ is given by

$$\langle k \rangle = p(N-1) = \frac{a}{N^z}(N-1). \tag{1}$$

For the following parameter values we have

(i) $a = 0.5, z = 1$;

$$\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} \frac{0.5}{N}(N-1) = 0.5. \tag{2}$$

The network does not contain a giant component.

(ii) $a = 2, z = 1$;

$$\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} \frac{2}{N}(N-1) = 2. \tag{3}$$

The network contains a giant component.

(iii) $a > 0,\ z = 2$;

$$\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} \frac{a}{N^2}(N-1) = 0. \tag{4}$$

The network does not contain a giant component.

(iv) $a > 0, z = 0.5$.

$$\lim_{N\to\infty} \langle k \rangle = \lim_{N\to\infty} \frac{a}{N^{1/2}}(N-1) = \infty. \tag{5}$$

The network contains a giant component.

(c) The average degree of a network in the $\mathbb{G}(N,p)$ ensemble is given by $\langle k \rangle = p(N-1)$. By inserting in this expression the value of p given by $p = a/N^z$, we get

$$\langle k \rangle = p(N-1) = \frac{a}{N^z}(N-1) \simeq \frac{a}{N^{z-1}} \tag{6}$$

where the last expression is valid for $N \gg 1$.
The limit for $N \to \infty$ this gives

$$\lim_{N\to\infty} \langle k \rangle = \begin{cases} 0 & \text{if} \quad z > 1 \\ a & \text{if} \quad z = 1, \\ \infty & \text{if} \quad z < 1. \end{cases} \tag{7}$$

(d) A random network is subcritical if $\langle k \rangle < 1$. Therefore the condition for a random network with $p = a/N^z$ to be subcritical in the limit $N \to \infty$ is given by

$$\langle k \rangle = \lim_{N\to\infty} \frac{a}{N^z}(N-1) = \lim_{N\to\infty} \frac{a}{N^{z-1}} < 1. \tag{8}$$

Therefore one of the two following conditions *(I)* or *(II)* must be met

(I) $\quad z > 1$
(II) $\quad z = 1 \quad \text{and} \quad a < 1.$

(e) A random network is supercritical if $\langle k \rangle > 1$. Therefore the condition for a random network with $p = a/N^z$ to be supercritical in the limit $N \to \infty$ is given by

$$\langle k \rangle = \lim_{N\to\infty} \frac{a}{N^z}(N-1) = \lim_{N\to\infty} \frac{a}{N^{z-1}} > 1. \tag{9}$$

Therefore one of the two following conditions *(III)* or *(IV)* must be met

$$\begin{array}{ll} \textit{(III)} & z < 1 \\ \textit{(IV)} & z = 1 \quad \text{and} \quad a > 1. \end{array}$$

(f) A random network is critical if $\langle k \rangle = 1$. Therefore the condition for a random network with $p = a/N^z$ to be critical in the limit $N \to \infty$ is given by

$$\langle k \rangle = \lim_{N\to\infty} \frac{a}{N^z}(N-1) = \lim_{N\to\infty} \frac{a}{N^{z-1}} = 1. \tag{10}$$

Therefore it must be at the same time

$$z = 1 \quad \text{and} \quad a = 1. \tag{11}$$

- **2. Random networks in the $\mathbb{G}(N,p)$ ensemble with $p = c/(N-1)$ where $c > 0$.**

  (a) Calculate the average number of triangles $\mathcal{N}_3^{\text{triangles}}$ in the network, by evaluating first the number of ways to select 3 nodes out of N nodes, and secondly the probability that the selected nodes are all connected to each other.

  (b) Show that in the limit $N \to \infty$ the average number of triangles in the network is

  $$\mathcal{N}_3^{\text{triangles}} \simeq \frac{1}{6}c^3. \tag{12}$$

  This means that the number of triangles is constant, neither growing or vanishing, in the limit of large N.

- *Notes on the solution*

  (a) The number of ways to choosing 3 nodes out of the N nodes of the network is given by $\begin{pmatrix} N \\ 3 \end{pmatrix}$. The probability that any three nodes of the network are all linked is given by p^3. Therefore we have that the average number of triangles $\mathcal{N}_3^{\text{triangles}}$ in a network of the $\mathbb{G}(N,p)$ ensemble with $p = \frac{c}{N-1}$ is given by

  $$\mathcal{N}_3^{\text{triangles}} = \begin{pmatrix} N \\ 3 \end{pmatrix} p^3.$$

  (b) In the large network limit, $N \to \infty$ we have:

  $$\begin{aligned} \mathcal{N}_3^{\text{triangles}} &= \begin{pmatrix} N \\ 3 \end{pmatrix} p^3 = \frac{N!}{3!(N-3)!}\left(\frac{c}{N-1}\right)^3 \\ &= \frac{1}{3!}\frac{N(N-1)(N-2)}{(N-1)^3}c^3 = \frac{1}{6}c^3. \end{aligned} \tag{13}$$