Queen Mary
University of London

Complex Networks (MTH6142)
Solutions of Formative Assignment 4

- **1. Degree distribution of random graphs**
  A random graph ensemble $\mathbb{G}(N, p)$ with $p = \frac{c}{N-1}$ has a binomial degree distribution

$$P_B(k) = \left(\begin{array}{c} N-1 \\ k \end{array} \right) p^k (1-p)^{N-1-k}, \tag{1}$$

that in the limit of $N \gg 1$ can be approximated by a Poisson distribution $P_P(k)$ given by

$$P_P(k) = \frac{1}{k!} c^k e^{-c}. \tag{2}$$

(a) Calculate the generating function

$$G_B(x) = \sum_{k=0}^{N-1} P_B(k) x^k \tag{3}$$

for the binomial degree distribution $P_B(k)$ given by Eq. (1).

(b) Using the properties of the generating functions, evaluate the first moment $\langle k \rangle$ and the second moment $\langle k(k-1) \rangle$ of the degree distribution $P_B(k)$ given by Eq. (1).

(c) Calculate the generating function

$$G_P(x) = \sum_{k=0}^{\infty} P_P(k) x^k \tag{4}$$

for the Poisson degree distribution $P_P(k)$ given by Eq. (2).

(d) Using the properties of the generating functions, evaluate the first moment $\langle k \rangle$ and the second moment $\langle k(k-1) \rangle$ of the degree distribution $P_P(k)$ given by Eq. (2).

(e) Show that the first $\langle k \rangle$ and second moment $\langle k(k-1) \rangle$ of the binomial distribution $P_B(k)$ obtained in (b) are the same as the first $\langle k \rangle$ and second $\langle k(k-1) \rangle$ moments of the Poisson distribution $P_P(k)$ obtained in (d), as long as $p = \frac{c}{N-1}$ with c constant and $N \to \infty$.

- *Notes on the solution*

(a) The generating function $G_B(x)$ is given by

$$G_B(x) = \sum_{k=0}^{N-1} P_B(k)x^k = (1-p+px)^{N-1}. \tag{5}$$

(b) The first are second moment of the degree distribution $P_B(k)$ are given by

$$\begin{aligned} \langle k \rangle &= \left.\frac{dG_B(x)}{dx}\right|_{x=1} = p(N-1), \\ \langle k(k-1) \rangle &= \left.\frac{d^2G_B(x)}{dx^2}\right|_{x=1} = p^2(N-1)(N-2). \end{aligned} \tag{6}$$

(c) The generating function $G_P(x)$ is given by

$$G_P(x) = \sum_{k=0}^{\infty} P_P(x)x^k = e^{-c+cx}. \tag{7}$$

(d) The first and second moment of the Poisson degree distribution can be obtained by differentiating the generating function $G_P(x)$. In particular we have

$$\begin{aligned} \langle k \rangle &= \left.\frac{dG_B(x)}{dx}\right|_{x=1} = c \\ \langle k(k-1) \rangle &= \left.\frac{d^2G_B(x)}{dx^2}\right|_{x=1} = c^2. \end{aligned} \tag{8}$$

(e) Since we have $c = p(N-1)$ we have always

$$\langle k \rangle = p(N-1) = c. \tag{9}$$

Moreover in the large network limit $N \to \infty$ we have

$$\langle k(k-1) \rangle = p^2(N-1)(N-2) = \frac{c^2}{N^2}(N-1)(N-2) \to c^2. \tag{10}$$

Therefore the first and second moment of the binomial distribution $P_B(k)$ with $p = \frac{c}{N-1}$ in the large network limit $N \to \infty$, are the same as the first and second moment of the Poisson distribution $P_P(k)$ with average degree c.

- **2. A given random network**
  Consider a random network in the ensemble $\mathbb{G}(N,p)$ with $N = 4 \times 10^6$ nodes and a linking probability $p = 10^{-4}$.

  (a) Calculate the average degree $\langle k \rangle$ of this network.

  (b) Calculate the standard deviation σ_P using the approximated degree distribution given by Eq. (2).

  (c) Assume that you observe a node with degree 2×10^3. How many standard deviations is this observation from the mean? Is this an expected observation or is this an unexpected observation?

- *Notes on the solution*

  (a) The average degree of the network is $\langle k \rangle = p(N-1) = 10^{-4} \times (4 \times 10^6 - 1) \simeq 400$.

  (b) The degree distribution $P(k)$ can be approximated with a Poisson distribution with average degree $\langle k \rangle = c \simeq 400$, i.e.

  $$P(k) = \frac{1}{k!} c^k e^{-c} = \frac{1}{k!} (400)^k e^{-400}. \tag{11}$$

  The second moment $\langle k(k-1) \rangle = c^2$.
  The variance of the degree distribution is $\sigma_P^2 = \langle k(k-1) \rangle + \langle k \rangle - \langle k \rangle^2 = c$.
  The standard deviation is $\sigma_P = \sqrt{c} = \sqrt{p(N-1)} \simeq \sqrt{400)} = 20$

  (c) The observation is $k = 2 \times 10^3$ it is 80 stardard deviations away from the mean. In fact

  $$\frac{k - \langle k \rangle}{\sigma_P} \simeq \frac{2000 - 400}{20} = 80. \tag{12}$$

  Therefore this observation is unexpected.