Queen Mary University of London

# Complex Networks (MTH6142)
# Solutions of Formative Assignment 3

- **1. Centrality measures of a given directed network**
  Consider the adjacency matrix $\mathbf{A}$ of a directed network of size $N = 4$ given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In the following we will indicate with $\mathbf{1}$ the column vector $\mathbf{1}$ with elements $1_i = 1\ \forall i = 1, 2 \ldots, N$ and we will indicate with $\mathbf{I}$ the identity matrix.
a) Draw the network
b) Calculate the eigenvector centrality using its definition (Def 1 of Chapter 3). Is the result expected? (*Explain why*).
c) Calculate the Katz centrality

$$\mathbf{x} = \beta(\mathbf{I} - \alpha\mathbf{A})^{-1}\mathbf{1} = \beta\sum_{n=0}^{\infty}\alpha^n\mathbf{A}^n\mathbf{1}. \tag{1}$$

d) Calculate the PageRank centrality

$$\mathbf{x} = \beta(\mathbf{I} - \alpha\mathbf{A}\mathbf{D}^{-\mathbf{1}})^{-1}\mathbf{1} = \beta\sum_{n=0}^{\infty}\alpha^n[\mathbf{A}\mathbf{D}^{-\mathbf{1}}]^n\mathbf{1} \tag{2}$$

where $\mathbf{D}$ is a diagonal matrix with non-zero elements $D_{ii} = \kappa_i = \max(k_i^{out}, 1)$ and $k_i^{out}$ is the out-degree of node $i$.

- *Notes on the solution*
  a) The network is a directed network of $N = 4$ nodes and is it shown in Figure 1.
  b)The network described by the adjacency matrix $\mathbf{A}$ given by Eq. (1), is a directed network without any strongly connected components, therefore the eigenvector centrality is given by $x_i = 0\ \forall i = 1, 2, 3, 4$.
  To see how the iterative procedure for calculation $x_i^{(n)}$ work in this case we start with the "democratic ansatz" $x_i^{(0)} = 1/4\ \forall i = 1, 2, 3, 4$. Using

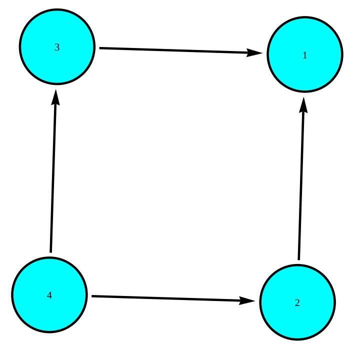


Figure 1: The graphical representation of the network with adjacency matrix **A** given by Eq. (1).

$\mathbf{x}^{(n)} = \mathbf{A}^n \mathbf{x}^{(0)}$, we obtain

$$\mathbf{x}^{(1)} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \\ 0 \end{pmatrix}$$

$$\mathbf{x}^{(2)} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x}^{(3)} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{4} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Therefore $\mathbf{x}^{(n)} = \mathbf{0}$ for $n \geq 3$. It follows that $\mathbf{x} = \mathbf{0}$.
c)The Katz centrality $\mathbf{x}$ can be calculated by

$$\mathbf{x} = \beta \left( \mathbb{I} - \alpha \mathbf{A} \right)^{-1} \mathbf{1} = \beta \sum_{n=0}^{\infty} \alpha^n \mathbf{A}^n \mathbf{1}. \tag{3}$$

Now we have by definition $\mathbf{A}^0 = \mathbb{I}$, moreover $\mathbf{A}^2$ and $\mathbf{A}^3$ are given by

$$\mathbf{A}^2 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad \mathbf{A}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and therefore $\mathbf{A}^n = 0$ for $n \geq 3$. Using the Eq. (3), we have therefore

$$\begin{aligned}
\mathbf{x} &= \beta \left[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \beta \begin{pmatrix} 1 + 2\alpha + 2\alpha^2 \\ 1 + \alpha \\ 1 + \alpha \\ 1 \end{pmatrix}.
\end{aligned}$$

d)The PageRank centrality $\mathbf{x}$ can be calculated by

$$\mathbf{x} = \beta \left( \mathbb{I} - \alpha \mathbf{A}\mathbf{D}^{-1} \right)^{-1} \mathbf{1} = \beta \sum_{n=0}^{\infty} \alpha^n \left( \mathbf{A}\mathbf{D}^{-1} \right)^n \mathbf{1}. \tag{4}$$

with $D_{ii} = \max(1, k_i^{out})$. Therefore we have

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \quad \mathbf{D}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Therefore the matrices $\left( \mathbf{A}\mathbf{D}^{-1} \right)^n$ are given by

$$\begin{aligned}
\mathbf{A}\mathbf{D}^{-1} &= \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
\left( \mathbf{A}\mathbf{D}^{-1} \right)^2 &= \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

and finally $\left( \mathbf{A}\mathbf{D}^{-1} \right)^n = 0$ for $n \geq 3$. Therefore the PageRank centrality is given by

$$\begin{aligned}
\mathbf{x} &= \beta \left[ \mathbb{I} + \alpha \mathbf{A}\mathbf{D}^{-1} + \alpha^2 \left( \mathbf{A}\mathbf{D}^{-1} \right)^2 \right] \mathbf{1} \\
&= \beta \left[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \alpha \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \beta \begin{pmatrix} 1 + 2\alpha + \alpha^2 \\ 1 + \frac{1}{2}\alpha \\ 1 + \frac{1}{2}\alpha \\ 1 \end{pmatrix}.
\end{aligned}$$

- **2. Degree centrality of a undirected network**
  Consider an undirected network with adjacency matrix $\mathbf{A}$. The degree centrality $x_i$ of a node $i$ is given by its degree $k_i$, i.e. $x_i = k_i$.
  Show that the degree centrality $\mathbf{x}$ of an undirected network satisfies the following equation

$$\mathbf{x} = \mathbf{A}\mathbf{D}^{-1}\mathbf{x}, \tag{5}$$

  where $\mathbf{D}$ is a diagonal matrix with non-zero elements $D_{ii} = \kappa_i = \max(k_i, 1)$.

- *Notes on the solution*
  The equation (5) can be written also as

$$x_i = \sum_{j=1}^{N} A_{ij} \frac{1}{\kappa_j} x_j, \tag{6}$$

  where $N$ is the total number of nodes of the network. By inserting the definition of degree centrality $x_i = k_i$ we obtain

$$k_i = \sum_{j=1}^{N} A_{ij} \frac{1}{\kappa_j} k_j. \tag{7}$$

  Now let consider the nodes $j$ of degree $k_j = 0$, the contribution of these nodes to the sum on the left hand side of Eq. (7) is zero and we can safely write Eq. (7) as

$$\begin{aligned} k_i &= \sum_{j=1,N|k_j>0} A_{ij} \frac{1}{k_j} k_j = \sum_{j=1,N|k_j>0} A_{ij} \\ &= \sum_{j=1}^{N} A_{ij}, \end{aligned} \tag{8}$$

  where the last equality is also obtained by considering that for the nodes $j$ of degree $k_j = 0$ we have $A_{ij} = 0 \ \forall i = 1, 2, \ldots, N$ and therefore they do not contribute to the sum $\sum_{j=1}^{N} A_{ij}$. Finally the relation found in Eq. (8) is nothing else than the definition of the degree of node $i$ and is always true, showing that the degree centrality of an undirected network always satisfies Eq. (5).