Actuarial Mathematics II
MTH5125

Multiple Decrement Models

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Plan

- Multiple decrement model - expressed in terms of multiple state model
- Multiple Decrement Tables (MDT)
- UDD
- Constant transition forces
- The Associated Single Decrement Table (ASDT)

Multiple Decrement Model

Multiple decrement models are extensions of standard mortality models with simultaneous several causes of decrement.

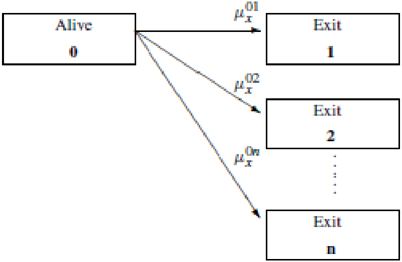
Examples include:

- life insurance contract is terminated because of death/survival or withdrawal (lapse).
- an insurance contract provides coverage for disability and death, which are considered distinct claims.
- life insurance contract pays a different benefit for different causes of death (e.g. accidental death benefits are doubled).
- pension plan provides benefit for death, disability, employment, termination and retirement.

Introduction Notation

A multiple decrement model is a multi-state model with:

- one active state (initial, State 0)
- one or more **absorbing** exit states.



Multiple Decrement Probabilities

I will introduce equivalent notations from the DHW textbook and the accepted (in practice) UK notations

- Dependent survival probability (): ${}_tp_x^{00}$ or ${}_t\left(ap\right)_x$
- Dependent transition probability: ${}_tp_x^{0j}$ or ${}_t\left(aq\right)_x^j$
- Dependent total transition probability: ${}_tp_x^{0\cdot}$ or ${}_t\left(aq\right)_x$
- Forces of transition: μ_{x+t}^{0j} or μ_{x+t}^{j}
- Total force of transition $\mu_{x+t}^{0\bullet}$ or $\left(a\mu\right)_{x+t}$
- Multiple decrement table
  - Active lives: l_x or $\left(al\right)_x$
  - Decrements: $d_x^{(j)}$ or $\left(ad\right)_x^{(j)}$

Multiple Decrement Example

A 10-year term insurance policy is issued to a life aged 50. The sum insured, payable immediately on death, is $\$200,000$ and premiums are payable continuously at a constant rate throughout the term. No benefit is payable if the policyholder lapses, that is, cancels the policy during the term. Calculate the annual premium rate using the following two sets of assumptions.

(a) The force of interest is 2.5% per year. The force of mortality is given by $\mu_x = 0.002 + 0.0005(x - 50)$.
No allowance is made for lapses.
No allowance is made for expenses.

(b) The force of interest is 2.5% per year.
The force of mortality is given by $\mu_x = 0.002 + 0.0005(x - 50)$.
The transition intensity for lapses is a constant equal to 0.05.
No allowance is made for expenses.

Multiple Decrement Example

(a) Since lapses are ignored an appropriate model is the alive-dead model

The annual premium rate P is calculated from equivalence principle:

$$P\bar{a}^{00}_{50:\overline{10}|} = 200,000P\bar{A}^{01}_{50:\overline{10}|}$$

Multiple Decrement Example

$$\bar{A}^{01}_{50:\overline{10}|} = \int_0^{10} e^{-\delta t}\ {}_tp^{00}_{50}\mu^{01}_{50+t}dt \text{ and } \bar{a}^{00}_{50:\overline{10}|} = \int_0^{10} e^{-\delta t}\ {}_tp^{00}_{50}dt$$

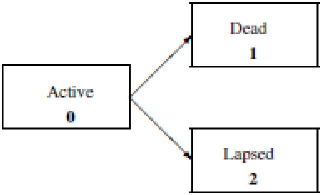
$$\begin{aligned}
{}_tp^{00}_{50} &= \exp\left(-\int_0^t \mu^{01}_{50+s}ds\right) \\
&= \exp\left(-\int_0^t (0.002 + 0.0005s)\, ds\right) \\
&= \exp\left(-\int_0^t (0.002 + 0.0005s))\, ds\right) \\
&= \exp\left(-0.002t - 0.00025t^2\right)
\end{aligned}$$

We can use numerical integration to calculate the integrals.

$$P = 200,000\frac{0.03807}{8.6961} = 875.49$$

Multiple Decrement Example

(b) Add: the transition intensity for lapses is a constant equal to 0.05
To allow for lapses, the model should be a double decrement model:



A single starting state and two exit states

Multiple Decrement Example

The annual premium rate P is calculated from equivalence principle:

$$P\bar{a}^{00}_{50:\overline{10}|} = 200,000P\bar{A}^{01}_{50:\overline{10}|}$$

It looks the same as before!

$$P = \frac{200,000P\bar{A}^{01}_{50:\overline{10}|}}{\bar{a}^{00}_{50:\overline{10}|}}$$

Multiple Decrement Example

But what is different?

$$
\begin{aligned}
{}_t p_{50}^{00} &= \exp\left(-\int_0^t \left(\mu_{50+s}^{01} + \mu_{50+s}^{02}\right) ds\right) \\
&= \exp\left(-\int_0^t (0.002 + 0.0005s + 0.05)\, ds\right) \\
&= \exp\left(-0.052t - 0.00025t^2\right)
\end{aligned}
$$

Apply numerical integration and get

$$P = 200,000 \frac{0.0289}{6.9269} = 834.54$$

Multiple Decrement Example

- The premium allowing for lapses is a lower
- The insurer will make a profit from any lapses:
  - not allowing for lapses, the policy value at any duration is positive and a lapse (with no benefit payable) releases this amount as profit to the insurer.
  - allowing for lapses, these profits can be used to reduce the premium.

Multiple Decrement Example

- In practice, the insurer may prefer not to allow for lapses when pricing policies if, as in this example, this leads to a higher premium.
- The decision to lapse is totally at the discretion of the policyholder on many factors, both personal and economic
- Where lapses are used to reduce the premium, the business is called lapse supported.
- Because lapses are unpredictable, lapse supported pricing is considered somewhat risky and has proved to be a controversial technique.

Multiple Decrement Example

- The choice of model depends on the terms of the policy and on the assumptions made by the insurer.
- The two models used in this example are clearly different, but they are connected.
- The difference is that the later model has more exit states;
  - The probability that the policyholder, starting at age 50, 'dies', that is enters state 1, before age $50 + t$ is different for the two models.

First model:

$$\begin{aligned} {}_{10}p_{50}^{01} &= \int_0^{10} {}_tp_{50}^{00}\mu_{50+t}^{01} dt \\ &= \int_0^{10} \exp\left(-0.002t - 0.00025t^2\right)(0.002 + 0.0005t)\,dt \\ &= 0.044 \end{aligned}$$

Multiple Decrement Example

Second model:

$$
\begin{aligned}
{}_{10}p_{50}^{01} &= \int_0^{10} {}_t p_{50}^{00} \mu_{50+t}^{01} dt \\
&= \int_0^{10} \exp\left(-0.052t - 0.00025t^2\right)(0.002 + 0.0005t)\, dt \\
&= 0.033
\end{aligned}
$$

in the insurance with lapses model.

Multiple Decrement Example

- The connections between the models are that **the single exit state in 'Dead' in the first model**, is **one of the exit states in the later model** and the transition intensity into this state, μ_x^{01}, is the same in the two models.
- The explanation of the differences of ${}_{10}p_{50}^{01}$ is that in the insurance with lapses model we are eliminating the policy holders that die after lapsing their policies - that is we interpret that 'Dead' as dying in the active state. If we increase the lapse decrement the probability of dying from active decreases, as more lives lapse before they die.

Dependent probabilities

We will denote **the dependent survival probability** with

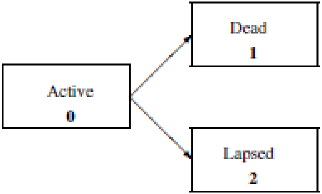
$$ {}_tp_x^{00} \equiv \ {}_t\left(ap\right)_x $$

probability of remaining in the active state when other risks of decrement are present. Note the two possible notations .
The **dependent transition probability**

$$ {}_tp_x^{0j} \equiv \ {}_t\left(aq\right)_x^j $$

One year probabilities are denoted as **rates**:
$p_x^{0j} \equiv \ (aq)_x^j$ is the dependent rate of decrement by decrement j at age $x-$ this is consistent with the terminology mortality for rate.
Second notation is the UK notation.

Multiple Decrement Table: One year relations



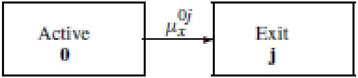
$$p_x^{00} + p_x^{01} + p_x^{02} = 1$$

or in the UK notation:

$$(ap)_x + (aq)_x^1 + (aq)_x^2 = 1$$

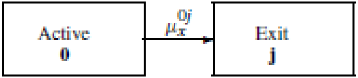
Multiple Decrement Example

When working with ,multiple decrement models we are often interested in a hypothetical simpler model with only one of the exit states $(0$ and $j)$ but with the transition intensity μ^{0j}_{x+t} taken from the full decrement model.
A simpler assumption: One exit state j with the same transition intensity into the state: independent single decrement model for decrement j



Independent Single Decrement Example

Independent single decrement model for decrement j



Reduced two state model with:
Independent survival probability denoted as:
$${}_tp_x^{*(j)} \equiv {}_tp_x^j = \exp\left(-\int_0^t \mu_{x+u}^{0j} du\right) \text{ and}$$

Independent transition probability: ${}_tq_x^{*(j)} \equiv {}_tq_x^j = \int_0^t {}_sp_x^{*(j)} \mu_{x+s}^{0j} ds$

Note that the second notation refers to the UK notation.

Similar to the life table functions l_x and d_x for the alive- dead model we will derive a multiple decrement table
Let l_{x_0} be the radix of the table at the initial age x_0
Define

$$l_{x+t} = l_{x_0}\ {}_tp_{x_0}^{00}$$

and for $j = 1, 2, ...m$ and $x \geq x_0$,

$$d_x^{(j)} = l_x p_x^{0j}$$

l_x - the expected number of active lives (in state 0) at age x out of l_{x_0}

$d_x^{(j)}$- the expected number of lives exiting by mode of decrement j in the year of age x to $x+1$

Multiple Decrement Table Example

An excerpt from a multiple decrement table for an insurance policy offering benefits on death or diagnosis of critical illness is given below. The insurance expires on the earliest event of death $(j = 1)$, surrender $(j = 2)$ and critical illness diagnosis $(j = 3)$.

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
40	10000	51	4784	44
41	95121	52	4526	47
42	90496	53	4268	50
43	86125	54	4010	53
44	82008	55	3753	56
45	78144	56	3496	59
46	74533	57	3239	62
47	71175	57	2983	65
48	68070	58	2729	67
49	65216	58	2476	69
50	62613	58	2226	70

Multiple Decrement Table Example

(a) Calculate ${}_3p_{45}^{00}$, p_{40}^{01}, ${}_5p_{41}^{03}$

$${}_3p_{45}^{00} = \frac{l_{48}}{l_{45}} = \frac{68070}{78144} = 0.87108$$

$$p_{40}^{01} = \frac{d_{40}^{(1)}}{l_{40}} = \frac{51}{10000} = 0.00051$$

$${}_5p_{41}^{03} = \frac{d_{41}^{(3)} + d_{42}^{(3)} + ... + d_{45}^{(3)}}{l_{41}} = 0.00279$$

Multiple Decrement Table Example

(b) Calculate the probability that a policy issued to a life aged 45 generates a claim for death or critical illness before age 47

$$ {}_2p_{45}^{01} + {}_2p_{45}^{03} = \frac{d_{45}^{(1)} + d_{46}^{(1)} + d_{45}^{(3)} + d_{46}^{(3)}}{l_{45}} = 0.00299 $$

(c) Calculate the probability that a policy issued to a life aged 40 is surrendered between age 45 and 47.

$$ {}_5p_{40}^{00}\ {}_2p_{45}^{02} = \frac{d_{45}^{(2)} + d_{46}^{(2)}}{l_{40}} = 0.06735 $$

Fractional age with UDD

UDD: here it means uniform distribution of decrements
For $0 \leq t \leq 1$ and integer x and for each exit mode j assume that for $j \neq 0$:

$$ {}_t p_x^{0j} = t \times p_x^{0j} $$

The exits from the starting state are uniformly spread over each year.

Fractional age with constant transition forces

For $0 \leq t \leq 1$ and integer x assume that for each exit mode j, μ_{x+t}^{0j} is a constant for each age x and equal to $\mu^{0j}(x)$:

$$\mu^{0\bullet}(x) = \sum_{k=1}^{m} \mu^{0k}(x)$$

$\mu^{0\bullet}(x)$ is the total force of transition out of state 0 at age $x + t$ for $0 \leq t < 1$ and:

$$_{t}p_{x}^{00} = \exp\left(-\mu^{0\bullet}(x)\, t\right)$$

Note that the **total force of transition out of state** 0 for the year of age x to $x + 1$

$$\begin{aligned} p_x^{0\bullet} &= 1 - p_x^{00} = \sum_{k=1}^{m} p_x^{0k} \\ &= 1 - \exp\left(-\mu^{0\bullet}(x)\right) \end{aligned}$$

Constant transition forces

Then for integer x and for $0 \leq t < 1$, we have:

$$ {}_{t}p_x^{0j} = \frac{p_x^{0j}}{p_x^{0\bullet}} \left(1 - \left(p_x^{00} \right)^t \right) \tag{9.2} $$

we can also use:

$$ {}_{t}p_x^{0j} = \frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} \left(1 - e^{-t\mu^{0\bullet}(x)} \right) $$

Constant transition forces

Proof

$$ {}_{t}p_{x}^{0j} = \int_{0}^{t} {}_{r}p_{x}^{00} \mu_{x+r}^{0j} dr = \int_{0}^{t} e^{-r\mu^{0\bullet}(x)} \mu^{0j}(x)\, dr $$

by constant transition forces, hence:

$$ {}_{t}p_{x}^{0j} = \frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} \left(1 - e^{-t\mu^{0\bullet}(x)}\right) = \frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} \left(1 - \left(p_{x}^{00}\right)^{t}\right) $$

Constant transition forces

Let $t \rightarrow 1 : \frac{\mu^{0j}(x)}{\mu^{0\bullet}(x)} = \frac{p_x^{0j}}{p_x^{0\bullet}}$ and hence:

$$ {}_t p_x^{0j} = \frac{p_x^{0j}}{p_x^{0\bullet}} \left(1 - \left(p_x^{00}\right)^t\right) $$

Intuition:
$\left(1 - \left(p_x^{00}\right)^t\right)$ is the total probability of exit under the constant transition force assumption and $\frac{p_x^{0j}}{p_x^{0\bullet}}$ divides this exit probability into the different decrements in proportion to the full one year exit probabilities

Example

Calculate ${}_{0.2}p_{50}^{0j}$ for $j = 1, 2, 3$ using the table provided earlier and assuming (a) UDD in all decrements between integer ages and (b) constant transition forces in all decrements between integer ages.

(a) UDD

$$
\begin{aligned}
{}_{0.2}p_{50}^{01} &= 0.2 \times p_{50}^{01} = 0.000185 \\
{}_{0.2}p_{50}^{02} &= 0.2 \times p_{50}^{02} = 0.007110 \\
{}_{0.2}p_{50}^{03} &= 0.2 \times p_{50}^{03} = 0.000224
\end{aligned}
$$

Example

(b) Under constant transition forces

$$ {}_{0.2}p_{50}^{0j} = \frac{p_{50}^{0j}}{p_x^{0\bullet}} \left(1 - \left(p_{50}^{00}\right)^{0.2}\right) $$

$p_x^{0\bullet} = 1 - p_x^{00}$ and

$$ \begin{aligned} {}_{0.2}p_{50}^{01} &= 0.000188 \\ {}_{0.2}p_{50}^{02} &= 0.007220 \\ {}_{0.2}p_{50}^{03} &= 0.000227 \end{aligned} $$

Deriving independent rates

Note that the decrement j independent survival probability is:

$$p_x^{*(j)} = \exp\left(-\int_0^1 \mu_{x+t}^{0j} dt\right)$$

or

$$p_x^{*(j)} = \exp\left(\log\left(p_x^{00}\right)^{\frac{p_x^{0j}}{p_x^{0\bullet}}}\right)$$

Hence:

$$p_x^{*(j)} = \left(p_x^{00}\right)^{\frac{p_x^{0j}}{p_x^{0\bullet}}} \tag{9.8}$$

Note that the UK notation is p_x^j

The associated single decrement table:

For each of the causes of decrement in an MDT, a single-decrement table can be defined showing the operation of that decrement independent of the others.
This is called the associated single-decrement table (ASDT)
Each table represents a group of lives reduced continuously by only one decrement.
The associated probabilities in the ASDT are **the independent rates** or **absolute rates of decrements**