Actuarial Mathematics II
MTH5125

Multiple State Models - Reserves

Dr. Melania Nica

Spring Term

Policy values

When we consider the reserves that need to be held for an insurance policy more general than life insurance, e.g. disability insurance, it is clear that a different reserve needs to be held, depending on the state currently occupied.
For example, under disability insurance:

- if the life is currently healthy, it is certain that they are currently paying premiums and possible that they might, in future, receive benefits.
- if the life is currently sick, it is certain that they are currently receiving benefits and possible that they might, in future, resume paying premiums.

Policy values

The state j policy value time t,denoted ${}_tV^{(j)}$ is the expected value at time t of the future loss random variable for a policy which is in State j at time t.

$$\begin{aligned} {}_tV^{(j)} &= EPV \text{ at } t \text{ of future benefits } + \text{expenses} \\ & -EPV \text{ at } t \text{ of future premiums,} \\ & \text{given the insured is in state } j \text{ at } t \end{aligned}$$

Policy values-Example

Consider a 20-year disability income insurance policy issued to a healthy life aged 50. Benefits of \$60,000 per year are payable continuously while the life is sick; a death benefit of \$30,000 is payable immediately on death. Premiums of \$5,000 per year are payable continuously while the life is healthy. Note that recovery from sick to healthy is possible.

Calculate ${}_{10}V^{(0)}$ and ${}_{10}V^{(1)}$ using the Standard Sickness Death Table with interest at 5% per year.

Policy values-Example

At time 10 the policy holder is aged 60 and the remaining term of the policy is 10 years.
For a policyholder in state 0 at time $t = 10$, the EPV of future benefits minus the EPV of future premiums is:

$$\begin{aligned} {}_{10}V^{(0)} &= 60,000\bar{a}^{01}_{60:\overline{10}|} + 30,000\bar{A}^{02}_{60:\overline{10}|} - 5,000\bar{a}^{00}_{60:\overline{10}|} \\ &= 60,000\,(0.6476) + 30,000\,(0.16382) - 5,000\,(6.5885) \\ &= 10,829 \end{aligned}$$

Policy values-Example

For a policy holder in State 1 at time 10 :

$$
\begin{aligned}
{}_{10}V^{(1)} &= 60,000\bar{a}^{11}_{60:\overline{10}|} + 30,000\bar{A}^{12}_{60:\overline{10}|} - 5,000\bar{a}^{10}_{60:\overline{10}|} \\
&= 60,000\,(6.9476) + 30,000\,(0.21143) - 5,000\,(0.0670) \\
&= 422,862
\end{aligned}
$$

Note that we must allow for the EPV of future premiums ($5,000\bar{a}^{10}_{60:\overline{10}|}$) as there is a possibility that the policy holder will move back into state 0 and resume paying premiums.

Policy values: Recursions - Thiele's differential equation

- Define $V^i(t)$ to be the policy value, on a given valuation basis, in respect of a life in state $i \in S$ at time t.
- For all i states, an annuity-type benefit payable continuously at rate $b_i(t)$ per annum if the life is in state i at time t.
- For all distinct i, j, a sum assured of $b_{ij}(t)$ payable immediately on a transition from state i to state j at time t.
- A force of interest $\delta(t)$.
- A complete set of transition intensities μ_{x+t}^{ij} for all distinct i, j.
- Possibly expenses payable continuously at rate $e_i(t)$ per annum if in state i at time t, or as a lump sum $e_{ij}(t)$ on transition from state i to state j at time t. Analogous to the benefits $b_i(t)$ and $b_{ij}(t)$

Policy values

Suppose the life to be in state i at time t (i.e. age $x+t$) and asking, what happens in the next time dt?

(1) The reserve currently held is equal to the policy value $V^i(t)$, by definition.

(2) In time dt, interest of $V^i(t)\delta(t)\,dt$ will be earned by these assets.

(3) In time dt, a cashflow of $b_i(t)dt$ will be paid by the office.

Policy values

(4) For each state $j \neq i$, a transition to state j may occur, with probability $\mu^{ij}_{x+t} dt$. If it does, the following happens:

– the sum assured $b_{ij}(t)$ is paid;

– the reserve necessary while in state j, equal to the policy value $V^j(t)$, must be set up;

– the reserve being held, $V^i(t)$, is available to offset these costs.

- the expected cost of a transition into state j is therefore:$\mu^{ij}(x+t)\, dt(b_{ij}(t) + V^j(t) - V^i(t))$.

Policy values

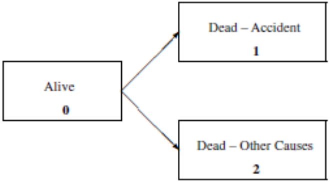
The general form of Thiele's equations:

$$\frac{d}{dt}V^i(t) = V^i(t)\delta(t) - b_i(t)\sum_{j\neq i}\mu_{x+t}^{ij}\left(b_{ij}(t) + V^j(t) - V^i(t)\right)$$

Note that this is a system of simultaneous ODEs, one for each state i. If, as is usually the case, benefits and force of interest do not depend on t, we get a simpler system:

$$\frac{d}{dt}V^i(t) = V^i(t)\delta - b_i\sum_{j\neq i}\mu_{x+t}^{ij}\left(b_{ij} + V^j(t) - V^i(t)\right)$$

Consider the accidental death model illustrated in Figure 8.2.



Let $\mu_x^{01} = 10^{-5}$ and $\mu_x^{02} = A + Bc^x$ for all x where $A = 5 \times 10^{-4}$, $B = 7.6 \times 10^{-5}$ and $c = 1.09$.

(a) Calculate

(i) ${}_{10}p_{30}^{00}$,

(ii) ${}_{10}p_{30}^{01}$, and

(iii) ${}_{10}p_{30}^{02}$.

(b) An insurance company uses the model above to calculate premiums for a special 10-year term life insurance policy. The basic sum insured is $\$100,000$, but the death benefit doubles to $\$200,000$ if death occurs as a result of an accident. The death benefit is payable immediately on death. Premiums are payable continuously throughout the term. Using an effective rate of interest of 5% per year and ignoring expenses, for a policy issued to a life aged 30

(i) calculate the annual premium for this policy, and

(ii) calculate the policy value at time 5 in state 0.

Policy values-Example

(a)-(i)

$$
\begin{aligned}
{}_{10}p_{30}^{00} &= \exp\left(-\int_0^{10}\left(\mu_{30+t}^{01}+\mu_{30+t}^{02}\right)dt\right)\\
&= \exp\left(-10^{-4}-\int_0^{10}\left(A+Bc^{30+t}\right)dt\right)\\
&= \exp\left(-10^{-4}\right)s^{10}g^{c^{30}\left(c^{10}-1\right)}
\end{aligned}
$$

where $s=\exp\left(-A\right)=0.999490$,
$g=\exp\left(-B/\log c\right)=0.999118$
Hence

$$
{}_{10}p_{30}^{00}=0.979122
$$

Policy values-Example

(a)-(ii)

$$ {}_{10}p_{30}^{02} = \int_{0}^{10} {}_{t}p_{30}^{00} \mu_{30+t}^{02} dt = 0.20779 $$

This can be evaluated by numerical integration - we use Repeated Simpson's Rule (Appendix B) in the DHW
(a)-(iii)

$$ {}_{10}p_{30}^{01} = 1 - {}_{10}\, p_{30}^{00} - {}_{10}\, p_{30}^{02} = 0.000099 $$

Policy values-Example

Numerical integration: Repeated Simpson's Rule

This rule is based on Simpson's rule which gives the following approximation:

$\int_a^{a+2h} f(x)\,dx \approx \frac{h}{3}\left(f(a) + 4f(a+h) + f(a+2h)\right)$

This approximation arises by approximating the function f by a quadratic function that goes through the three points $(a, f(a))$, $(a+h, f(a+h))$ and $(a+2h, f(a+2h))$.

Numerical integration: Repeated Simpson's Rule

Repeated application of this result leads to the repeated Simpson's rule, namely

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(a) + 4\sum_{j=1}^{n} f(a + (2j-1)h) + 2\sum_{j=1}^{n-1} f(a + 2jh) + f(b)\right)$$

Policy values-Example

Numerical integration: Repeated Simpson's Rule
If $n = 3$

$$4\sum_{j=1}^{3} f\left(a+(2j-1)h\right) + 2\sum_{j=1}^{2} f\left(a+2jh\right) =$$
$$4f\left(a+h\right) + 4f\left(a+3h\right) + 4f\left(a+5h\right) + 2f\left(a+2h\right) + 2f\left(a+4h\right)$$

$$\int_a^b f(x)\,dx = \frac{h}{3}\begin{pmatrix} f(a) + 4f(a+h) + 2f(a+2h) + \\ 4f(a+3h) + 2f(a+4h) + 4f(a+5h) + f(b) \end{pmatrix}$$

Please check the Excel file on QMPlus how we calculated ${}_{10}p_{30}^{01}$

Policy values-Example

(b) An insurance company uses the model above to calculate premiums for a special 10-year term life insurance policy. The basic sum insured is \$100,000, but the death benefit doubles to \$200,000 if death occurs as a result of an accident. The death benefit is payable immediately on death. Premiums are payable continuously throughout the term. Using an effective rate of interest of 5% per year and ignoring expenses, for a policy issued to a life aged 30

(i) calculate the annual premium for this policy, and

(ii) calculate the policy value at time 5 in state 0.

Policy values-Example

(b) -(i) The EPV for the premium P per year payable continuously is:

$$P\bar{a}^{00}_{30:\overline{10}|} = P\int_0^{10} v^t \ {}_tp^{00}_{30}\,dt$$

The EPV of the death benefit is:

$$\begin{aligned} & 200,000\bar{A}^{01}_{30:\overline{10}|} + 100,000\bar{A}^{02}_{30:\overline{10}|} \\ = \ & 200,000\int_0^{10} v^t \ {}_tp^{00}_{30}\mu^{01}_{30+t}\,dt + 100,000\int_0^{10} v^t \ {}_tp^{00}_{30}\mu^{02}_{30+t}\,dt \end{aligned}$$

Policy values-Example

$$P = \frac{200,000\bar{A}^{01}_{30:\overline{10}|} + 100,000\bar{A}^{02}_{30:\overline{10}|}}{\bar{a}^{00}_{30:\overline{10}|}} = 206.28$$

The integrals are evaluated using numerical integration, giving:
$P = 206.28$

Policy values-Example

The policy value at time 5 in state 0 is given by:

$$
\begin{aligned}
{}_5V^0 &= 100,000\bar{A}^{02}_{35:\overline{5}|} + 200,000\bar{A}^{01}_{35:\overline{5}|} - P\bar{a}^{00}_{35:\overline{5}|} \\
&= 167.15
\end{aligned}
$$